The Lorenz Attractor and Bifurcations

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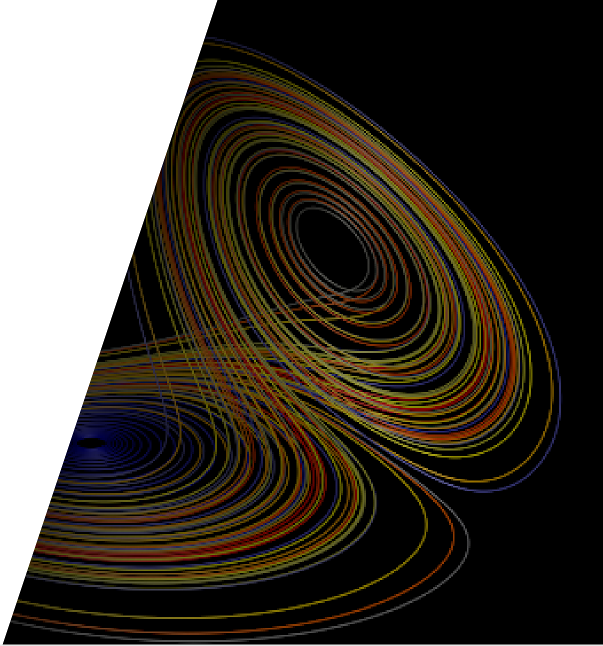

Agenda

Introduction

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Definition (Flow)

Let the *flow* of a system of differential equations be denoted $\phi(t; \mathbf{x})$, which we treat as a continuous dynamical system. In this case, we write $\phi(t; \mathbf{x}) = \phi^t(\mathbf{x})$ to mean the evolution of a point $\mathbf{x} \in \mathbb{R}^n$ after time t of the differential equation.

Can flows of autonomous ODEs exhibit chaotic behavior in their attracting sets?

Examples of Flows

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Suppose that we have the differential equation $\dot{x} = 3|x|^{2/3}$. What is $\phi^t(x)$?

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Example

What is the flow of a first-order linear differential equation, $\dot{\mathbf{x}} = A\mathbf{x}$?

$$\phi^t(\mathbf{x}) = e^{At}\mathbf{x}.$$

Here, note that the multiplier e^{At} is a matrix exponential.

Poincaré-Bendixson Theorem

Theorem ([Tes12])

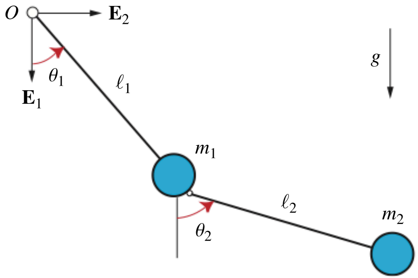
Given a differentiable autonomous dynamical system defined on an open subset of $\mathbb{R}^2$, every non-empty compact ω-limit set of an orbit, which contains only finitely many fixed points, is either

- *a fixed point,*
- *a periodic orbit, or*
- *a connected set composed of a finite number of fixed points, together with homoclinic and heteroclinic orbits connecting these.*

Here, autonomous means without a time-dependent driving force. For example, for constants $k, \alpha > 0$, we would write that $\ddot{x} = -kx$ is autonomous, but $\ddot{x} = -kx + \sin \alpha t$ is not.

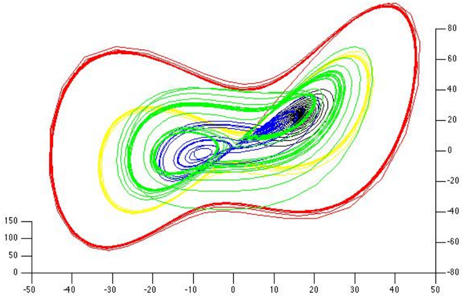
Chaotic Attractors

- We can't have chaotic behavior in attracting sets for 1-dimensional or 2-dimensional autonomous dynamical systems on the reals.
- Anti-example: double pendulum is **four**-dimensional, state $\langle \theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2 \rangle$.



Chaotic Attractors

- The *Lorenz system* was the first such example of chaos, a 3-dimensional nonlinear system of differential equations with complicated behavior in the limit set.



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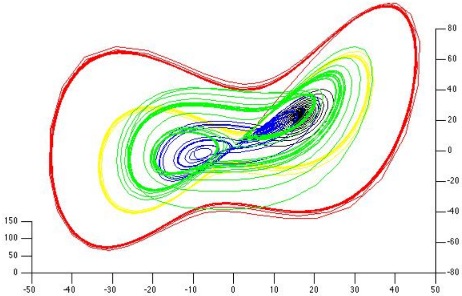

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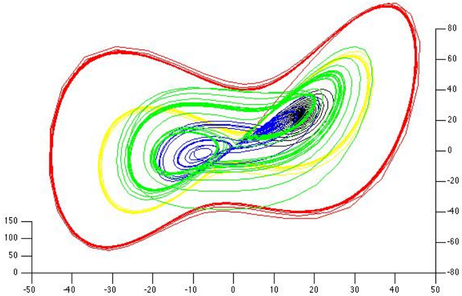
- Lorenz modeled convection in the atmosphere.
- Ueda also found a periodically forced nonlinear oscillator in two dimensions that had complicated dynamics [Ued91].



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The Lorenz System

The *Lorenz system* is a system of three nonlinear ordinary differential equations, given by

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy. \end{aligned}$$

Here, σ, r, and b are three positive parameters.[1] We focus on the special case where $\sigma = 10$ and $b = 8/3$, but we will vary the value of r. Lorenz originally studied $r = 28$ when modeling convection in the atmosphere.

[1]Some sources replace r by ρ and b by β.

Fixed Points of the Lorenz System

By examining the differential equation, fixed points must satisfy $x = y$, $x(r - 1 - z) = 0$, and $bz = x^2$. In other words, we can just look at the behavior in the x–z plot, projected from the plane $y = x$. There are up to three fixed points:

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- Otherwise, assume that $z = r - 1$. For another fixed point, we need $bz = x^2$, so $r > 1$. Then, we have two other fixed points, at $x = y = \pm\sqrt{bz}$ and $z = r - 1$.

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So when $r > 1$, our fixed points are

$$(0, 0, 0), (\sqrt{bz}, \sqrt{bz}, r - 1), \text{and } (-\sqrt{bz}, -\sqrt{bz}, r - 1).$$

Repelling Origin

Next let's look at the behavior at the origin. The eigenvalues of the Jacobian at the origin are

$$\lambda_s = -b < 0,$$

$$\lambda_{ss} = \frac{-(\sigma+1) - \sqrt{(\sigma+1)^2 + 4\sigma(r-1)}}{2},$$

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- When $r > 1$, the origin is a fixed point of saddle type, with dimension-2 stable manifold and dimension-1 unstable manifold (unstable!).

Other Fixed Points

- The other two fixed points when $r > 1$ are called $\mathbf{P}^+$ and $\mathbf{P}^-$.
- The characteristic polynomial of the Jacobian is a cubic, and one of the eigenvalues is always guaranteed to be a negative real number. This corresponds to the normal to the Lorenz attractor (a 2D manifold).
- Specifically, the characteristic polynomial is

$$p_r(\lambda) = \lambda^3 + \lambda^2(\sigma + b + 1) + \lambda b(r + \sigma) + 2b\sigma(r - 1) = 0.$$

- Luckily, this is a pretty simple cubic. Fixing $\sigma = 10$ and $b = 8/3$, we can graph this in λ and see how the roots change as we vary r.

Other Fixed Points (cont.)

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  - When $r = r_1$, the complex eigenvalues are pure imaginary, and they have zero real part. This means that there is some "center manifold" where the system attracts. This is known as a subcritical Hopf bifurcation, as the stability of the system changes, and there momentarily exists an unstable periodic orbit.

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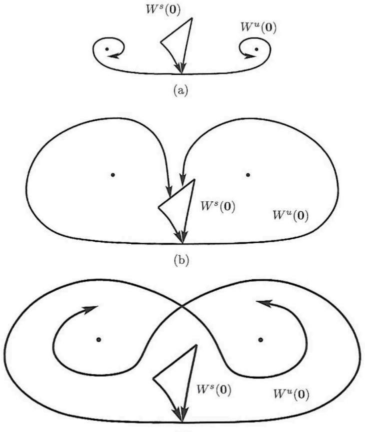
  - When $r = 28$, the system demonstrates chaotic behavior [Tuc99].
- Sage code is available at `https://tinyurl.com/sage-lorenz`.

Global Bifurcations

- Another interesting behavior is that there exists a homoclinic bifurcation.



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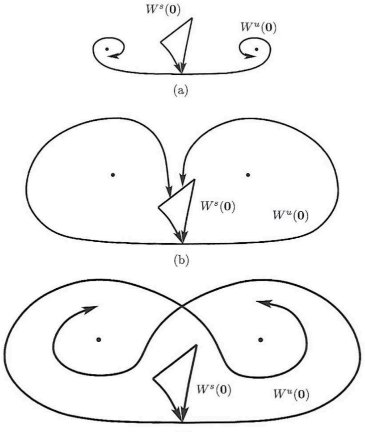

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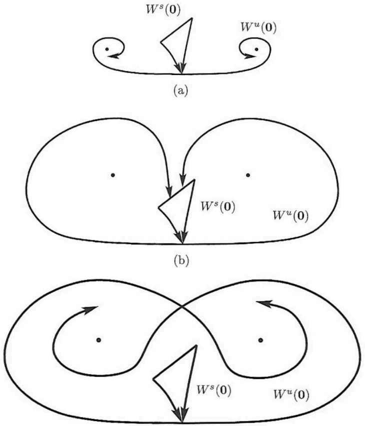
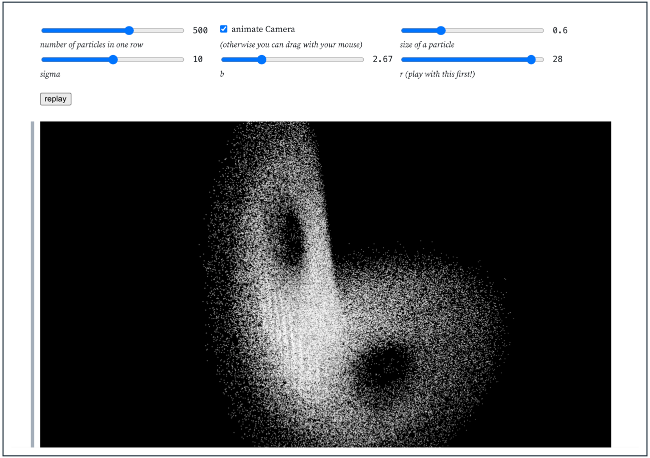
- If you start at the origin and follow the unstable manifold trajectory outward, you'll eventually end up attracting to a limit cycle in one of the two "lobes" when r is small.
- As r increases, the limit cycle grows until it intersects with the saddle point at the origin for $r = r_0 \approx 13.926$. This is called a *homoclinic orbit* [VA98].



Discussion: Bifurcation Experiments



https://observablehq.com/@ekzhang/lorenz-system

Agenda

Limit Sets of Flows

Definition (ω-limit set)

Given a flow ϕ and an initial point $\mathbf{x} \in \mathbb{R}^n$, we define the ω-*limit set* of $\mathbf{x}$ by

$$\omega(\mathbf{x}) = \bigcap_{n \in \mathbb{N}} \mathrm{cl}(\{\phi^t(\mathbf{x}) : t \geq n\}),$$

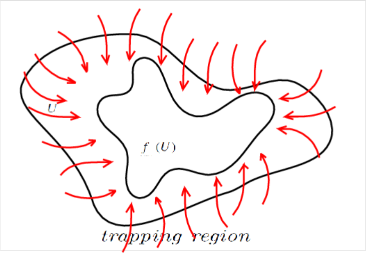
where $\mathrm{cl}(S)$ denotes the closure of a set S. This is the set of points that the orbit of $\mathbf{x}$ gets arbitrarily close to, in the forward limit.

How does this definition differ from $\omega(x)$ for a *discrete* dynamical system?

Trapping Regions

Definition (Trapping region)

A *trapping region* for a flow $\phi(t;\mathbf{x})$ is a compact set $\mathbf{U}$ such that $\phi(t;\mathbf{U}) \subset \text{int}(U)$ for all $t > 0$.



Trapping Regions (cont.)

Theorem

The boundary of a trapping region U *is moved a positive distance into its interior for any positive amount of time,* $t > 0$.

Trapping Regions (cont.)

Theorem

The boundary of a trapping region $\mathbf{U}$ is moved a positive distance into its interior for any positive amount of time, $t > 0$.

Proof.

The boundary of a compact set is compact, so we can apply the extreme value theorem. For any $t > 0$, $\phi^t : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, so

$$\min_{\mathbf{x} \in \mathrm{bd}(\mathbf{U})} \left\{ \min_{\mathbf{y} \in \mathbf{U}} \|\mathbf{y} - \phi^t(\mathbf{x})\| \right\} > 0.$$

□

Quantifying an "Energy" Function

Definition (Lyapunov function)

A *Lyapunov function* for an autonomous dynamical system $\dot{\mathbf{y}} = g(\mathbf{y})$ is a $\mathcal{C}^1$ function $L : \mathbb{R}^n \to \mathbb{R}$ satisfying $-\nabla L \cdot g > 0$ for some region of $\mathbb{R}^n$.

This definition is by convention, and it is engineered so that $L(\phi^t(\mathbf{y}_0))$ is monotonically decreasing as t advances. (Why?)

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This definition is by convention, and it is engineered so that $L(\phi^t(\mathbf{y}_0))$ is monotonically decreasing as t advances. (Why?)
Solution: Just use the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t} L(\phi^t(\mathbf{y}_0)) = \nabla L \cdot \frac{\mathrm{d}\phi^t(\mathbf{y}_0)}{\mathrm{d}t} = \nabla L \cdot g < 0.$$

Lyapunov Functions in Practice

- Using this definition, we can define a trapping region using a test function L as the set
$$\mathbf{U} = L^{-1}((-\infty, C]) = \{\mathbf{x} : L(\mathbf{x}) \leq C\},$$
which is closed because preimages of continuous functions preserve open/closed sets.
- If the gradient of L is nonzero on $L^{-1}(C)$, then we also get that $\mathrm{int}(\mathbf{U}) = L^{-1}((-\infty, C))$.

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which is closed because preimages of continuous functions preserve open/closed sets.
- If the gradient of L is nonzero on $L^{-1}(C)$, then we also get that $\mathrm{int}(\mathbf{U}) = L^{-1}((-\infty, C))$.
- **Question:** What if we only have $-\nabla L \cdot g > 0$ for some region **outside** a compact set $\mathbf{U}$, i.e., where $L(\mathbf{x}) > C$? Is it still good enough?

Attractors

Definition (Attracting set)

A set $\mathbf{A}$ is called an *attracting set* for the trapping region $\mathbf{U}$ if

$$\mathbf{A} = \bigcap_{t \geq 0} \phi^t(\mathbf{U}).$$

Equivalently, $\mathbf{A}$ is the largest invariant set contained in $\mathbf{U}$.

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Definition (Attractor)

An attracting set $\mathbf{A}$ is called an *attractor* if there is no proper subset $0 \neq \mathbf{A}' \subsetneq \mathbf{A}$ that is also an attracting set.

Lorenz Attractor

Theorem. The trajectories for the Lorenz system tend to a set of zero volume.

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Proof.

The change in volume of an orbit over time is equal to the surface integral of the vector field $\mathbf{F}$. By the divergence theorem, this is also equal to the integral of $\nabla \cdot \mathbf{F}$ over the volume. For the Lorenz system, we have $\nabla \cdot \mathcal{F} = -\sigma - 1 - b$, which is a negative constant, so

$$\dot{V}(t) = -(\sigma - 1 - b)\, V(t) \implies V(t) = e^{-(\sigma+1+b)t}\, V(0).$$

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Since the limit volume cannot be infinite (which can be shown using a very coarse Lyapunov function, bounding by an ellipsoid), we conclude that the volume of the limit set must be zero, since $V(t)$ decays exponentially. $\square$

Agenda

Mathematica Notebook

- Pay attention to the track of the ball in the animation. Try to see if it is possible to predict the specific trajectories of the ball at different parameters.
- Specifically, try to find the bifurcation values brought up earlier in the presentation and model what happens to the ball in these scenarios.
- Finally, try to make sense of the rates of change in the trajectory as it relates to the three central equations governing the Lorenz System.
- Feel free to play around with the Lorenz equations. For example, many applications use the Lorenz system but with the first equation reading $\dot{x} = -\sigma(x - y - A\sin(\omega t))$, where ω is between 0 and 1.

Agenda

History of the Lorenz System

- Fascinated with meteorology as a child, he spent his post-college life working as a weather forecaster in the Army Air Corps for World War II.
- Advent of the computer encouraged him to build weather simulations.
- Originally had set of 12 differential equations with 12 variables, but eventually cut it down to the three differential equations we see today, the Lorenz system.

Edward Lorenz (1917-2008)

An Interesting Story

- Lorenz was running his simulations many times, realizing that the weather patterns were seemingly changing in a pseudo-periodic orbit without ever repeating the same conditions again.
- He decided to start the computer at a specific weather sequence to get a better view of it. However, when he started the simulation, the result gave completely different weather patterns than he had experienced in previous simulations.
- The error occurred because he inputted the initial conditions with three significant figures when the computer was working in six. This minute difference caused the entire system to change and created the basis for chaotic systems today.

The Lorenz System of Atmospheric Flow

In the Lorenz system (equations posted here again for reference), we can attempt to explain some of the variables and constants:

$$\dot{x} = -\sigma x + \sigma y, \quad \dot{y} = rx - y - xz, \quad \dot{z} = -bz + xy.$$

- σ represents the ratio of fluid viscosity to thermal conductivity. Essentially the ratio of how quickly fluid flows through the system to how effective it absorbs heat in contact with other molecules (related to Prandtl Number).
- r represents the difference in temperature between the top and bottom of the atmospheric column (related to Rayleigh Number).
- b simply describes the bounds of our system, more specifically the ratio of the width to height (2D cross-section).

The Lorenz System of Atmospheric Flow

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$$\dot{x} = -\sigma x + \sigma y, \quad \dot{y} = rx - y - xz, \quad \dot{z} = -bz + xy.$$

- x, y, z all vary dynamically in time. x is the convective flow (motion of heat through the atmosphere as hot air rises). Positive x pertains to clockwise motion.
- y is the horizontal temperature distribution, found by taking the difference between ascending and descending currents. Positive y pertains to warmer currents on the right (correlates strongly with positive x)
- z is the vertical temperature distribution.

Deriving the Intuition Behind r

The main source of heat to the surface of Earth is via the Sun (internal energy from the Earth heats the surface much less). To find r, we must first define the flux of Sun into Earth's orbit ($\frac{W}{m^2}$), given by

$$F_0 = \frac{L}{4\pi d^2},$$

where L is the luminosity of the Sun, in watts, and d is the Earth's orbital semi-major axis.

Continuing Derivation of r

Now the amount that the Earth absorbs, in power (W), is:

$$P_{in} = \pi r^2 (1 - A) F_0,$$

where A is the planetary albedo and r is the radius of the Earth.

Continuing Derivation of r

The *Stefan-Boltzmann law* tells us that the radiant flux emitted from the Earth is

$$F_1 = \sigma T^4,$$

where T is the temperature of Earth and $\sigma = 5.67 * 10^{-8}\, W/m^2/K^4$ (Stefan's constant). The total radiation that Earth gives out is this flux/area times the surface area of the earth, which is just $4\pi r^2$. Setting $P_{in} = P_{out}$ to satisfy conservation of energy, we get the equation:

$$\pi r^2 (1 - A) F_0 = 4\pi r^2 \sigma T^4 \implies \sigma T^4 = \frac{1}{4}(1 - A) F_0.$$

Continuing Derivation of r

- Solving the equation in the previous slide for T, using the appropriate values for L, d, σ, r and approximating Albedo to be 0.3 on Earth, we get that the equilibrium temperature on Earth should be around 256 Kelvin.
- However, we see that the average temperature on Earth is actually 288 Kelvin (15 degrees Celsius). While a small amount of the error can be explained by the assumptions surrounding Stefan's Constant, the majority of the error comes from the presence of Greenhouse Gasses in the atmosphere.
- These gasses warm up the surface, trapping air in a convective cycle within the troposphere. Thus we expect r to be $288 - 256$ Kelvin, or 32. Ignoring smaller sources of error between observational and empirical data that is beyond the scope of this presentation, we see that the assumption of $r = 28$ is reasonable.

Significance of Lorenz in Atmospheric Dynamics

- Allows us to see how different emissions spread by convection through the atmosphere.
- Allows us to model weather and wind more accurately.
- Allows us to model the effect of anthropogenic emissions on vertical and horizontal temperature gradients.

Agenda

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