Math 132: Differential Topology

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Abstract

These are notes for Harvard's *Math 132*, a class on differential topology, as taught by Joe Harris[1] in Spring 2021. We will cover most of the textbook *Differential Topology* by Guillemin and Pollack [GP10], taking the approach of manifolds embedded in the reals. We also briefly describe the modern definition of manifolds and see applications to algebraic topology.

Course description: Differential manifolds, smooth maps and transversality. Winding numbers, vector fields, index and degree. Differential forms, Stokes' theorem, introduction to cohomology.

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[1] With teaching fellows: Serina Hu, Alec Sun, Gaurav Goel

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1 January 25th, 2021

This is the first lecture of the course. Joe Harris introduces the main topics. Broadly, we will cover differential topology from a geometric perspective grounded in $\mathbb{R}^n$, focusing particularly on the study of *geometric objects* rather than abstract concepts like topological spaces. The recommended prerequisites for this class are *at least one of* Math 131 and Math 122, and we'll use basic material from both. Category theory is intentionally not taught in this class.

1.1 Diffeomorphisms

Differential topology focuses on the set of topological spaces **Top**, equipped with some notion of *smooth* mapping between them. Since we are specifically considering subsets of $\mathbb{R}^n$ in this class (rather than a more general notion of topological space), we can immediately write down the following concrete definitions.

Definition 1.1 (Smooth map)**.** A function $f : X \to Y$ where $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, is called *smooth*, or equivalently, $\mathcal{C}^\infty$, if it is infinitely differentiable.

We have to be a little careful with the above definition, since X will usually not be an open subset in the cases we consider. To remedy this technically, we also call f smooth if for every point $x \in X$, there exists an extension of f to some neighborhood $U \subset \mathbb{R}^n$ of x that is smooth on U, and agrees with f on $U \cap X$.

Definition 1.2 (Diffeomorphism)**.** A function $f : X \to Y$ is called a *diffeomorphism* if it is a smooth bijection with smooth inverse. In other words, there exists $g : Y \to X$ such that f, g are both $\mathcal{C}^\infty$, and $f \circ g = \mathrm{id}_Y$, $g \circ f = \mathrm{id}_X$.

We call $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ *diffeomorphic* if there exists a diffeomorphism $f : X \to Y$. Since it is transitive and symmetric, diffeomorphism is an equivalence relation on subsets of the reals. In particular, it is a stronger equivalence relation than homeomorphism, as there is more structure.

Example 1.3. A 2-sphere is not homeomorphic to a 2-torus, as shown by their respective fundamental groups. Therefore, it is also not diffeomorphic to a 2-torus.

Example 1.4. Although a sphere and a cube are homeomorphic, they are not diffeomorphic, since the cube has sharp corners at its vertices and edges.

This pair of examples should provide a pretty good intuition about diffeomorphism. Not only does it preserve topological structure, but it also preserves **differential structure**, so jagged surfaces must stay jagged, while smooth surfaces must stay smooth.

1.2 Differential Manifolds

So far, we've been working with arbitrary subsets of $\mathbb{R}^n$. Here we'll try to narrow our focus a bit, and only study certain nice subsets that locally "look" like the reals.

Definition 1.5 (Manifold)**.** A subset $X \subset \mathbb{R}^n$ is called a *manifold* with dimension k if for all $p \in X$, there exists a neighborhood U of p such that U is diffeomorphic to $\mathbb{R}^k$, or equivalently, that U is diffeomorphic to an open ball $B_r(o)$ in $\mathbb{R}^k$.[2] We notate this as $U \cong \mathbb{R}^k \cong B_r(o)$.

[2] Here, we need to be careful with the metric used to define an open ball. This is somewhat context dependent, and you can't just use any norm like in topological spaces. The Euclidean norm works fine here, but some other norms that don't arise from an inner product might not work, as you'll end up with non-diffeomorphic balls.

A manifold of dimension k is also sometimes called a *k-manifold.* Manifolds will be the primary object of study in this course, as they tend to be really nice to work with. Here's a useful tool that we'll introduce to deal with manifolds.

Definition 1.6 (Coordinate chart)**.** A collection of functions $f : U \to \mathbb{R}^k$ is called a *coordinate chart*, or *atlas*, if it associates to every point p an open neighborhood $p \in U$, and f is a diffeomorphism $q \mapsto (x_1(q), x_2(q), \ldots, x_k(q))$. Each of these x_j is called a *coordinate function.*

Note that we can prove that a subset of $\mathbb{R}^n$ is a manifold by finding an appropriate atlas. For example, any arc of the unit circle S^1 is trivially diffeomorphic to the open ball $B^1 \cong \mathbb{R}$ by taking for instance $\theta \mapsto \arctan\theta$. Therefore, two arcs of length $\pi + \epsilon$ are enough to cover the unit circle with an atlas, which shows that it is a 1-manifold.

Note. Harris remarks that a lot of "old-fashioned" spherical geometry and trigonometry deals with spherical coordinates on the globe. Spherical coordinates can also be viewed as a particular choice of coordinate chart on S^2.

Example 1.7. An open ball is trivially a manifold. Any n-sphere S^n is a manifold, as well as the torus S^2, and all polyhedra. However, a closed interval is not a manifold, as the no neighborhood of its endpoints is homeomorphic to an open ball. We'll discuss how to think about objects like closed intervals as a so-called *manifold with boundary* later on.

This concludes the first lecture. Next time, we'll talk more about the derivative!

2 January 27th, 2021

Today we discuss derivatives and tangent spaces. This lecture will develop some of the language that we'll need to talk about the geometric objects we're interested in, which should set the direction for future lectures.

2.1 The Derivative

Recall the univariate derivative from Calculus, as well as the multivariate partial derivative. These are useful primitives, but often require rooting ourselves in a coordinate system in $\mathbb{R}^n$. In this section, we introduce a more *geometric* definition of the derivative on manifolds.

Definition 2.1 (Derivative in $\mathbb{R}^n$)**.** Suppose that we have a smooth function $f : U \to \mathbb{R}^m$ with domain given by an open set $U \subset \mathbb{R}^n$. For any vector $h \in \mathbb{R}^n$, the *derivative* of f in direction h, taken at point $x \in U$, is defined by the limit

$$\mathrm{d}f_x(h) = \lim_{t\to 0} \frac{f(x+th) - f(x)}{t}.$$

In this case, the derivative of a function at a point, $\mathrm{d}f_x$, is a linear function $\mathbb{R}^n \to \mathbb{R}^m$. It can also be written compactly as the $m \times n$ Jacobian matrix

$$\mathrm{d}f_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}.$$

The key fact of derivatives as linear approximations is that they can be composed, using the chain rule. It turns out that the chain rule also applies in this definition.

Proposition 2.2 (Chain rule)**.** *If $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, and $W \subset \mathbb{R}^\ell$ are open sets, and we have smooth functions $f : U \to V$ and $g : V \to W$, such that $h = g \circ f$, then*

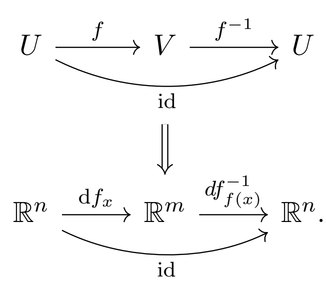
$$\mathrm{d}h_x = \mathrm{d}g_{f(x)} \circ \mathrm{d}f_x.$$

This compositionality indicates that the derivative can be viewed as part of a functor.

What about the case when f is a diffeomorphism in a neighborhood of the point x? In this case, we should expect that the derivative is an injective linear map (isomorphism), as otherwise, one or more dimensions would be "squashed" as part of the nullspace of the Jacobian.

Proposition 2.3. *If $f : U \to V$ is a diffeomorphism at $x \in U$, then $\mathrm{d}f_x$ is an isomorphism.*

Proof. Consider $f^{-1} : V \to U$. By the chain rule, we have the following diagram:



Therefore, we have shown that $\mathrm{d}f_x$ is invertible, as desired. □

It turns out that the converse of this statement is also true in a local sense, although it is much less obvious! In fact, it's such an important theorem that it has its own name.

Proposition 2.4 (Inverse function theorem)**.** *Given a smooth map $f : U \to V$ between $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ such that $\mathrm{d}f_x$ is invertible, there exists a neighborhood $U_0 \subset U$ of x that maps diffeomorphically onto its image under f.*

We should be explicit about this idea of a *local diffeomorphism.* Here is a motivating example.

Example 2.5. Consider $f : \mathbb{R} \to S^1 \subset \mathbb{R}^2$ given by $t \mapsto (\cos t, \sin t)$. This map is clearly not injective, as it wraps the real line around the circle an infinite number of times, so it is not a diffeomorphism.[3] However, it is a local diffeomorphism (see covering space for more on this).

Joe pauses here to make a point that in this course, we usually work in the category of differential manifolds, so the word "map" will occasionally be used to mean "smooth map" implicitly. Similarly, there is the notion of a *topological manifold*, which is similar to Definition 1.5 except with the word "diffeomorphism" replaced by "homeomorphism."

2.2 Tangent Spaces

Before we can extend Definition 2.1 to arbitrary manifolds, we need to invent the concept of a tangent space. This idea is motivated by the idea of linear approximation.

Definition 2.6 (Tangent space)**.** Given a k-manifold $X \subset \mathbb{R}^N$ and point $x \in X$, we can parameterize the manifold locally by taking a neighborhood U of x and diffeomorphism $\phi : \mathbb{R}^k \to U$ such that $\phi(0) = x$. This has derivative $\mathrm{d}\phi_0 : \mathbb{R}^k \to \mathbb{R}^N$ at 0. Then, the *tangent space of X at x*, denoted $T_x(X)$ is defined as the image of $\mathrm{d}\phi_0$, which is a linear subspace of $\mathbb{R}^N$.

You can visualize the tangent space intuitively as a k-space best approximating the surface of the manifold at that point, except *translated to include the origin.* For example, if X is a curve, then the tangent space at a point is a line parallel to the tangent line. If X is a surface, it becomes a plane parallel to the tangent plane. To be fully clear about this, we have the following.

Definition 2.7 (Geometric tangent space)**.** Given a manifold $X \subset \mathbb{R}^N$, we also define the *geometric tangent space* to X at x by the subset $x + T_x(X) \subset \mathbb{R}^N$.

We have to be careful though. There are many possible ways of drawing a coordinate chart for a manifold, and all of those could yield different functions $\mathrm{d}\phi_0$. How do we know if our definition of tangent space makes sense, if the coordinate chart is not unique? The beauty here is that the tangent space has a *universal property*; it is the same no matter which coordinates you choose.

Exercise 2.1. Show that Definition 2.6 is valid, i.e., that given two different local parameterizations $\phi : \mathbb{R}^k \to U$ and $\psi : \mathbb{R}^k \to U$ centered at x, we have

$$\operatorname{im} \mathrm{d}\phi_0 = \operatorname{im} \mathrm{d}\psi_0.$$

Proof. Since both of the maps are diffeomorphisms, we have the commutative triangle:

$$\begin{array}{ccc} \mathbb{R}^k & \underset{\phi^{-1}\circ\psi}{\overset{\psi^{-1}\circ\phi}{\rightleftarrows}} & \mathbb{R}^k. \\ & {\scriptstyle\phi}\searrow \qquad \swarrow{\scriptstyle\psi} & \\ & U & \end{array}$$

The result follows after taking the derivative and applying the chain rule. ∎

[3] Joe pictures this colorfully as an "infinite parking garage."

Our definition of the tangent space may feel a bit frustrating, as it seems like it is only a property of the manifold X and point $x \in X$. Why do we need this extraneous information of a coordinate chart, which we just throw away after constructing the definition? It seems unnecessary to add this extra degree of freedom that we need to prove is redundant. Joe mentions that there are ways of setting up the definitions so that the tangent space is an abstract vector space, but this is not the approach taken by our textbook, as it prefers to be concrete.

Moving on, now that we know what a tangent space is, we can define the derivative more generally, for smooth functions between manifolds. This is just what we want!

Definition 2.8 (Derivative for manifolds)**.** Suppose that we have a smooth function $f : X \to Y$ between manifolds, and a point $x \in X$. The *derivative* of f at x, denoted $\mathrm{d}f_x : T_x(X) \to T_y(Y)$, where $y = f(x)$, is the unique linear map between tangent spaces at x and y such that the chain rule holds. Formally, given coordinate charts $\phi : U \to X$ and $\psi : V \to Y$, let $h = \psi^{-1} \circ f \circ \phi$. Then, the derivative is defined as

$$\mathrm{d}f_x = \mathrm{d}\psi_0 \circ \mathrm{d}h_0 \circ \mathrm{d}\phi_0^{-1},$$

such that the following diagram commutes:

$$\begin{array}{ccc} T_x(X) & \xrightarrow{\mathrm{d}f_x} & T_y(Y) \\ {\scriptstyle \mathrm{d}\phi_0}\uparrow & & \uparrow{\scriptstyle \mathrm{d}\psi_0} \\ \mathbb{R}^k & \xrightarrow[\mathrm{d}h_0]{} & \mathbb{R}^k. \end{array}$$

We can check that this definition of $\mathrm{d}f_x$ does not depend on choice of ϕ and ψ.

You should feel like these definitions are unsatisfying and a bit messy, but it's an argument that everyone has to work through at least once to understand the derivative. If we focus on the goal here, what Definition 2.8 lets us do is write down a function $\mathrm{d}f_x$ that describes, given any local movement on the manifold X near x, in which direction the value of $f(x)$ changes on Y.

Our ultimate goal in this course is to **classify all kinds of maps** between manifolds. There are two ways of doing this: first in terms of their local behavior at a single point, and second in terms of global maps between manifolds with certain properties. In our next lecture, we will start tackling this problem by beginning our discussion of *immersions*, or maps where the source manifold dimension is strictly less than that of the target.

3 February 1st, 2021

Today we continue discussing the inverse function theorem and use it to analyze immersions. As Gaurav summarizes: "The whole point of differential topology is to reduce smooth maps (difficult) to linear algebra (easy), at least locally."

3.1 Smoothness and Differentiability

On a brief mathematical note, we deal primarily with $\mathcal{C}^\infty$ functions in this course, even though many of our theorems will also work with less strict requirements, such as $\mathcal{C}^1$ and $\mathcal{C}^k$ functions. But for convenience's sake,[4] most of the functions we deal with in practice are $\mathcal{C}^\infty$, so we'll just use this term interchangeably with "smooth" and "differentiable" functions.

Example 3.1 (Smooth functions)**.** All polynomials, arithmetic, exponents, exponentials, logarithms, trigonometric functions, and compositions or inverses of the above are $\mathcal{C}^\infty$ where defined. They are also holomorphic on their domain in the complex plane.

Basically, most non-piecewise functions you can write down will automatically be smooth.

3.2 Local Equivalence

We start by reiterating Proposition 2.4, as it is the key fact about derivatives that we'll be using today. If the derivative map $\mathrm{d}f_x$ is invertible, then the function $f : X \to Y$ is locally a diffeomorphism around x. The big picture is that if we have a map between manifolds of the same dimension, then there is an open set where the map is a local diffeomorphism, but there's another subset of points where it is not, and the derivative is singular. We'll come back to this idea in a couple lectures.

Example 3.2. Consider projecting the unit circle S^1 onto the horizontal axis. This map is a local diffeomorphism everywhere except at the points $(-1, 0)$ and $(1, 0)$, where it is two-to-one in a small neighborhood.

Example 3.3. Consider creasing a sheet of paper into an upside-down "V" shape (like a small tent), then projecting it down onto the table. This map is a local diffeomorphism everywhere except at the crease.

Example 3.4 (Complex exponent)**.** Consider the map $f : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$. This map essentially wraps the unit circle around itself in a two-to-one fashion. At every point of the source $\mathbb{C}$, this map is a local diffeomorphism, *except at the origin* where it is two-to-one.

In the first two examples, the submanifold where the map failed to be a local diffeomorphism had dimension one lower than the domain manifold. We therefore say that it has *codimension* 1. However, in the third example, the submanifold is a point in a plane and has *codimension* 2.

Definition 3.5 (Equivalent maps)**.** Suppose that we have two maps $f : X \to Y$ and $f' : X' \to Y'$. We say that f and f' are *equivalent maps* if there exist diffeomorphisms $\phi : X' \to X$ and $\psi : Y' \to Y$ such that the following square commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi\uparrow & & \uparrow\psi \\ X' & \xrightarrow{f'} & Y'. \end{array}$$

[4] In particular, Joe Harris says that dealing with $\mathcal{C}^k$ functions with finite k is "a pain in the a**."

Completely equivalent maps do not occur very often in practice, since they are entirely the same up to being expressed on different manifolds. However, the local version is very useful.

Definition 3.6 (Locally equivalent maps)**.** We say that $f : X \to Y$ and $f' : X' \to Y'$ are *locally equivalent* at $x \in X$, $x' \in X'$ if there exist open neighborhoods $x \in U$ and $x' \in U'$ such that $f|_U$ is equivalent to $f'|_{U'}$.

For an example of how to use the definition of local equivalence in practice, the next corollary follows directly from Proposition 2.4.

Corollary 3.6.1. *If $f : X \to Y$ and $f' : X' \to Y'$ are smooth maps between manifolds of equal dimension n, and there exist $x \in X$, $x' \in X'$ such that $\mathrm{d}f_x$ and $\mathrm{d}f'_{x'}$ are both isomorphisms, then f and f' are locally equivalent at x and x'.*

3.3 Immersions

So far, we've been looking at smooth maps $f : X \to Y$ between manifolds where $\dim X = \dim Y$. The natural nice property for these maps between same-dimension manifolds was for f to be a diffeomorphism. However, what if the manifolds are of different dimensions? In the case when $k = \dim X < \dim Y = \ell$, the next best thing is for f to be injective. This motivates the following definition.

Definition 3.7 (Immersion)**.** We say that a map $f : X \to Y$ where $\dim X \leq \dim Y$ is an *immersion* at point $x \in X$ if $\mathrm{d}f_x$ is injective.

For completeness, we'll also introduce the analogous definition in the other direction.

Definition 3.8 (Submersion)**.** We say that a map $f : X \to Y$ where $\dim X \geq \dim Y$ is a *submersion* at point $x \in X$ if $\mathrm{d}f_x$ is surjective.

It turns out that in either of these cases, we can give an explicit characterization of the local behavior of f near x, in local coordinates. The main theorem is an analogue of the inverse function theorem for immersions and submersions.

Proposition 3.9 (Local immersion theorem)**.** *Suppose that $f : X \to Y$ is a smooth map between manifolds X and Y, with dimensions k and ℓ, respectively. Suppose that we have some $x \in X$ and $y = f(x)$. Then, if f is an immersion at x, there exist local coordinates $x_1, \ldots, x_k$ in X and $y_1, \ldots, y_\ell$ in Y such that*

$$f(x_1, \ldots, x_k) = (x_1, \ldots, x_k, \underbrace{0, 0, \ldots, 0}_{\ell - k \text{ zeros}}).$$

In other words, any immersion around a point is locally equivalent to the "canonical immersion" of a vector subspace within another vector subspace.

Proof. Our main tool is the inverse function theorem. We just need to figure out how to transform the problem to apply it, since our manifolds X and Y might have different dimensions $k \neq \ell$. First, observe that if f is an immersion at x, then $\mathrm{d}f_x : T_x(x) \hookrightarrow T_y(y)$ is an inclusion by definition. Then, arbitrarily choose a complementing vector space $W \subset T_y(Y)$ such that[5]

$$T_y(Y) = \mathrm{d}f_x(T_x(X)) \oplus W.$$

[5] If we gave $T_y(Y)$ the additional structure of an inner product, then we could provide a canonical choice of orthogonal complement. However, for the purposes of proving this theorem, we can still pick some arbitrary complementary vector subspace for the zero elements; it doesn't matter which one.

With this new complementary space, consider the map

$$\begin{aligned} \bar{f} &: U \times W \to Y \\ &: (u, w) \mapsto (f(w), w). \end{aligned}$$

By the inverse function theorem, $\bar{f}$ is a local diffeomorphism, and therefore, f is an inclusion. $\square$

Note. For any function f, the set of $x \in X$ for which f is an immersion at x is an open subset. This is because the matrix of partial derivatives consists of smooth functions, and therefore its determinant is also a smooth (and in particular, $\mathcal{C}^0$) function.

This is a pretty nice statement. Let's now shift gears slightly to talk about global properties.

Definition 3.10 (Global immersion)**.** We say that a map $f : X \to Y$ where $\dim X \leq \dim Y$ is an *immersion* globally if it is a local immersion at all $x \in X$.

Clearly, Proposition 3.9 tells us that the image of an immersion is locally a submanifold. In other words, if f is an immersion at x, then there exists an open neighborhood $U \subset X$ of x such that $f(U) \hookrightarrow Y$ is a submanifold. However, a trickier question we can ask is this: when is the image of a global immersion $f : X \to Y$ a submanifold of Y?

Example 3.11. Let's show a few examples of immersions where the image is **not a submanifold** of the codomain, to illustrate some of the issues we might face.

1. Consider a map $f : S^1 \to \mathbb{R}^2$ that takes the circle to a figure-8 shape. Although this is locally an immersion, the image is not a submanifold of $\mathbb{R}^2$ because of the point where the two lobes of the "8" intersect. The problem in this case is that f is not one-to-one.

2. What if we constrain f to be injective? It turns out that this isn't enough. Consider an open interval $(0, 1)$ in the real line, which we map once again to the figure-8 shape, except we very slightly omit the track by putting both interval endpoints at the cross point! Now the map is injective and an immersion, but the image is the same, still not a submanifold.

3. Consider a map $f : \mathbb{R} \to T$, where $T = \mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$ is the torus. The way we construct this map is by placing or "lifting" $\mathbb{R}$ as a straight line through the origin in $\mathbb{R}^2$, then projecting it back down to the torus by modding out the lattice $\mathbb{Z}^2$.

   This map $\mathbb{R} \hookrightarrow \mathbb{R}^2 \twoheadrightarrow T$ is clearly an immersion, as both parts are immersions, but what does it look like? In the case when the slope of the line is a rational number, the line will hit a lattice point, so we wind around the torus a finite number of times, forming a knot shape. However, when the slope is irrational, the line never comes back to itself, and the image is actually dense in T. This is definitely not a submanifold.

These three examples should be fairly motivating; we need some additional criteria for the image of an immersion to be a submanifold. It turns out that all we need for this to hold is related to compactness! The problem in examples #2 and #3 is related to how the source manifolds $(0, 1)$, $\mathbb{R}$, are not compact. This lets us get arbitrarily close to any point without actually reaching it, which messes up the topology.

Definition 3.12 (Proper map)**.** We call a map $f : X \to Y$ *proper* if for all compact subsets $K \subset Y$, the preimage $f^{-1}(K)$ is compact.

Proposition 3.13. *If $f : X \to Y$ is an injective, proper immersion, then $f(X)$ is a submanifold.*

Proof. We omit the proof here, but Joe states that it can be motivated by our examples. □

Because it is a submanifold, we have a special name for this kind of map.

Definition 3.14 (Embedding)**.** An immersion $f : X \to Y$ is called an *embedding* if it is both injective and proper.

3.4 Submersions

We can write a direct analogue of Proposition 3.9 to the case of submersions.

Proposition 3.15 (Local submersion theorem)**.** *Suppose that $f : X \to Y$ is a smooth map between manifolds X and Y, with dimensions k and ℓ, respectively. Suppose that we have some $x \in X$ and $y = f(x)$. Then, if f is an submersion at x, there exist local coordinates $x_1, \dots, x_k$ in X and $y_1, \dots, y_\ell$ in Y such that*

$$f(x_1, \dots, x_k) = (x_1, \dots, x_\ell).$$

In other words, any submersion around a point is locally equivalent to a vector space projection.

Once again, we'll turn now to global properties.

Definition 3.16 (Global submersion)**.** We say that a map $f : X \to Y$ where $\dim X \geq \dim Y$ is an *submersion* globally if it is a local submersion at all $x \in X$.

Okay, so what's the analogue of our cool fact from earlier, Proposition 3.13? Our question will be, given a submersion $f : X \to Y$, in which cases will the fibers of f, i.e., the sets $f^{-1}(y) \subset X$ for points $y \in Y$, be submanifolds? It turns out this is just always the case, and we have something even stronger.

Definition 3.17 (Regular value)**.** If $f : X \to Y$ is a smooth map, then $y \in Y$ is called a *regular value* of f if f is a submersion at all points in $f^{-1}(y)$.

Proposition 3.18 (Preimage theorem)**.** *If $f : X \to Y$ is a smooth map, and y is a regular value of f, then $f^{-1}(y) \subset X$ is a submanifold.*

Proof. For all $x \in f^{-1}(y)$, there is an open neighborhood $U \subset X$ of x in the source that maps to Y as a linear projection of vector spaces. Therefore, $f^{-1}(y)$ is diffeomorphic to the kernel of this linear projection, so it must be a submanifold in U, and we conclude. □

Let's do one illustrative example to showcase this theorem.

Example 3.19 (S^n is a manifold)**.** Consider for some $k \geq 1$, the map $f : \mathbb{R}^k \to \mathbb{R}$ given by

$$f(x_1, \dots, x_k) = \sum_{i=1}^{k} x_i^2.$$

Notice that every $y \in \mathbb{R}$ is a regular value of f, except $y = 0$. Therefore, we immediately conclude that $f^{-1}(1) \cong S^{k-1}$ is a submanifold of $\mathbb{R}^k$.

Hence, the propositions from today can be added as tools to our toolbox, to prove that various subsets are manifolds. We'll see more of this next time!

4 February 3rd, 2021

Today we discuss a couple of fundamental ideas in algebraic topology, and in particular, the notion of *transversality*. Transversality is first introduced in §5 of [GP10], and we will continue investigating this concept for the rest of the semester.

4.1 More on Submersions

Recall that the notion of regular value from Definition 3.17 was useful in the preimage theorem. We introduce three new definitions to complement this one.

Definition 4.1 (Critical value)**.** If $f : X \to Y$ is a smooth map, then $y \in Y$ is called a *critical value* if it is not a regular value. In other words, there exists some $x \in f^{-1}(y)$ such that f is not a submersion at x.

Example 4.2. Consider a polynomial function $f : \mathbb{R} \to \mathbb{R}$. The critical values are precisely those y for which there exist x such that $f(x) = y$, and $f'(x) = 0$. In particular, local minima and maxima must all be critical values.

Definition 4.3 (Regular and critical points)**.** Given a map $f : X \to Y$, and a point $x \in X$, we call x a *regular point* if $\mathrm{d}f_x$ is surjective, and we call x a *critical point* if $\mathrm{d}f_x$ is not surjective.

Now we introduce an example, which will be representative of the type of situation that we'll be mostly studying in the future.

Example 4.4 (Doughnut through a slicer)**.** Consider the map $f : T \to \mathbb{R}$ from projecting a torus (embedded in $\mathbb{R}^3$) onto one of its non-rotational axes. The image of the torus is a closed interval, and the endpoints of the interval are critical values. There are two critical points corresponding to these, at a local minimum and maximum. There are also two more critical points at saddles (the "inner ring" of the torus), which also correspond to critical values.

What do the fibers of this map look like? This is the beginning of Morse theory, which we will talk about later in the semester. Initially, the fiber is just a single point, which turns into a circle. At the second critical value, the fiber becomes a figure-eight, where Proposition 3.18 fails. Then, in the middle of T, the fiber becomes a disjoint union of two copies of S^1, which is a manifold. In general, we have a change in the shape of the fiber at non-degenerate critical values.

Now we shift gears and talk about submersions again. Suppsose that we have a submersion $f : X \to Y$ that locally at $x \mapsto y$ looks like the map $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_\ell)$. Then we can view the maps $\mathrm{d}x_1, \ldots, \mathrm{d}x_\ell$ as independent functions from $T_x(X)$ to the real line $\mathbb{R}$, in the sense that their combination is surjective on the neighborhood. This motivates a definition and rephrasing.

Definition 4.5 (Independent functions)**.** A collection of smooth maps $g_1, \ldots, g_n : U \to \mathbb{R}$ are *independent* at x if their derivative functionals $\mathrm{d}(g_1)_x, \ldots, \mathrm{d}(g_\ell)_x$ are linearly independent.

The crux of independent functions is that if smooth, real-valued functions $g_1, \ldots, g_\ell$ are independent at each point where they all vanish, then the set of common zeros is a submanifold of X with codimension ℓ. This is just a restatement of Proposition 3.18, but it is very useful for proving that certain subsets of $\mathbb{R}^n$ are manifolds. Not only can we prove things are manifolds using the standard definition, by finding a coordinate chart, but we can also show this implicitly by writing down a set of independent functions whose zero-image describes it locally.

Example 4.6. Once again, we show that S^2 is a manifold, but this time with slightly new notation. Consider $g : \mathbb{R}^3 \to \mathbb{R}$ given by $g(x, y, z) = x^2 + y^2 + z^2$. Then, we have $\mathrm{d}g_{(x,y,z)} : \mathbb{R}^3 \to \mathbb{R}$ is given by the dual vector $(2x, 2y, 2z)$. The only critical point of g is at $(0, 0, 0)$, so the only critical value is at 0. Therefore,

$$g^{-1}(t) = \left\{(x, y, z) \mid x^2 + y^2 + z^2 = t\right\}$$

is a submanifold unless $t = 0$.

4.2 Introduction to Transversality

Transversality is a core topic in this course. You can view it as a generalization of both immersion and submersion, but for the purposes of this introduction, we will think of it as simply extending submersion. Our basic setup is to have a map $f : X^k \to Y^\ell$, where the superscripts indicate the dimensions of the manifolds, i.e., $k = \dim X$ and $\ell = \dim Y$. Consider a submanifold $Z \subset Y$, and look at $f^{-1}(Z)$. When is this preimage going to be a submanifold?

What we describe is a sufficient (but not necessary) condition for $f^{-1}(Z) \subset X$ being a submanifold, which provides a partial answer to the question. Roughly, consider Z as a common zero locus of a collection of independent functions.

Proposition 4.7. *If* $\operatorname{im} \mathrm{d}f_x + T_y(Z) = T_y(Y)$, *then* $f^{-1}(Z)$ *is a manifold at* x.

Proof. Let U be a neighborhood of $y \in Y$ such that locally, there exist independent functions $g_1, \ldots, g_\ell : U \to \mathbb{R}^\ell$ such that $Z \cap U = g^{-1}(0)$. Then, consider $g \circ f : V \to \mathbb{R}^\ell$. This composition of functions is a submersion at x because

$$\mathrm{d}(g \circ f)_x = \mathrm{d}g_y \circ \mathrm{d}f_x,$$

and $\operatorname{im} \mathrm{d}f_x + T_y(Z) = T_y(Y)$, but $T_y(Z) = \ker \mathrm{d}g_y$. □

Definition 4.8 (Transversality)**.** We say that a map f is *transverse* to Z if $\operatorname{im} \mathrm{d}f_x + T_y(Z) = T_y(Y)$ everywhere, for all $x \in f^{-1}(Z)$. We write this as $f \pitchfork Z$.

The simplest case is with an embedding $f : X \hookrightarrow Y$, where $X, Z \subset Y$ are both submanifolds. Then, $f^{-1}(Z) = X \cap Z$. Transversality holds when for all $p \in X \cap Z$, $T_pX + T_pZ$ span T_yY. In this case, we have that $X \cap Z$ is a submanifold of Y. Roughly speaking, you can visualize this as X and Z intersecting at a nonzero angle, versus being tangent to one another.

5 February 8th, 2021

Last week, we introduced the notion of transversality. Today we cover more details about transversality, and we begin discussing open and closed families of maps.

5.1 More on Transversality

We previously introduced the general definition of transversality (Definition 4.8), notated as $f \pitchfork Z$. We will extend this definition with the special notation, $f \pitchfork_x Z$, meaning that f is transverse to Z at the particular point $x \in X$. One special case of transversality is when $Z = \{p\}$, a single point, in which case $f \pitchfork Z$ is equivalent to p being a *regular value* of f. Another special case is when $f : X \hookrightarrow Y$ is an embedding.

Definition 5.1 (Transversality, embedding case)**.** Consider two submanifolds $X, Z \subset Y$. We say that X is transverse to Z, or $X \pitchfork Z$ as submanifolds of Y, if the embedding $f : X \hookrightarrow Y$ satisfies $f \pitchfork Z$. In other words, at every point $p \in X \cap Z$, we have $T_pX + T_pZ = T_pY$.

This special definition is perhaps easier to think about. You can imagine that at each $p \in X \cap Z$, we can choose a local coordinate chart $x_1, \ldots, x_k$ in Y (up to diffeomorphism) such that Z contains the locus of $x_1, \ldots, x_{n-k}$, while X contains the locus of $x_{\ell+1}, \ldots, x_n$. Then, the intersection of X and Z is a submanifold of codimension $k + \ell$.

Example 5.2. Consider what happens when $n = 2$, and $k = \ell = 1$. The transverse way for two curves in $\mathbb{R}^2$ to intersect at p is if their two slopes at p are different. In other words, the direct sum of tangent spaces of the two curves at p spans the plane.

Example 5.3. If $n = 3$, and $k = \ell = 1$, then transversality is impossible. Two curves in $\mathbb{R}^3$ can never intersect transversely; they will always be tangent at intersections. In general, if $X, Z \subset Y$ satisfy $\dim X + \dim Z < \dim Y$, then they cannot intersect transversely. The key intuition is that when two submanifolds intersect transversely, we can make a slight deformation of either manifold and not topologically affect the intersection; this is not the case for tangency.

Example 5.4. If $n = 3$, $\ell = 2$, and $k = 1$, then transverse intersection of a curve and a surface in $\mathbb{R}^3$ consists of a single point. A non-transverse intersection might look like a parabola tangent to a plane at a point. If $\ell = k = 2$, then two surfaces in $\mathbb{R}^3$ intersect at a line. However, a non-transverse intersection could be the intersection of the tip of a paraboloid with a plane, which is still a submanifold, but has incorrect dimension. Another example is a cylinder lying on a plane, which intersects at a line (correct dimension) but is still not transverse.[6]

Example 5.5. Consider the three-dimensional graph of the function

$$z = \mathrm{Re}((x+iy)^n) = x^n - \binom{n}{2} x^{n-2} y^2 + \binom{n}{4} x^{n-4} y^4 - \cdots .$$

This value oscillates up and down in a circular motion, as you go around the origin. What is the intersection of this manifold with the xy-plane? It turns out that this is the union of n lines through the origin! Clearly, this is not transverse.

[6] Joe compares this to Tolstoy: "All happy families are alike; each unhappy family is unhappy in its own way."

5.2 Homotopy and Stability

Consider the question of what it means to write down a "small deformation" of a map $f : X \to Y$. This can be easily extended to a small deformation of submanifolds. Of course, there's a well-studied formalization of this idea in topology: it's the idea of *homotopy*!

Definition 5.6 (Homotopy of manifolds)**.** Given two smooth maps $f : X \to Y$ and $g : X \to Y$ between manifolds, we say that f and g are *homotopic* ($f \sim g$) if there exists a smooth function $F : [0, 1] \times X \to Y$ such that $F(0, x) = f(x)$ and $F(1, x) = g(x)$ for all $x \in X$.

We call the function $F : [0, 1] \times X \to Y$ in the above definition of homotopy a *family of maps*. This motivates the following definition, which will capture the essence of our prior observations about transversality.

Definition 5.7 (Stable property)**.** A property of a map $f : X \to Y$ is called *stable* if for all families of maps $F : I \times X \to Y$ where $F(0, x) = f(x)$, there exists an $\epsilon > 0$ such that for all $t < \epsilon$, the map F_t also has that property.

Example 5.8. Consider a manifold M of dimension 1 in $\mathbb{R}^2$. The property "M contains the origin" is not a stable property, as shifting a curve slightly would make its path miss the origin. Similarly, intersecting the x-axis is not a stable property, as the graph of $y = x^2$ has this property, while the graph of $y = x^2 + \epsilon$ does not.

Example 5.9. Consider the graph of $y = x^3$. The property of this graph intersecting the x-axis once is not stable, as if you rotated the graph slightly clockwise (by angle ϵ), it would intersect the x-axis three times rather than once.

Okay, so those are some counterexamples, but then what *is* stable? This is the punchline.

Proposition 5.10. *Transversality $f \pitchfork Z$ is a stable property of f.*

Proposition 5.11. *If X is compact, then immersion, submersion, and local diffeomorphism are all stable properties of a map $f : X \to Y$ between manifolds.*

Proof. The basic idea is that for a perturbed map f_ϵ, the derivative $\mathrm{d}(f_\epsilon)_x$ is very similar to the original derivative $\mathrm{d}f_x$, as it changes smoothly in the vector space of all linear transformations. The subset of all matrices of maximal rank is open, so locally the map must remain a submersion (or replace with other equivalent property). Finally, by applying the compactness of X, we can turn this local property into a global statement using the extreme value theorem, and we conclude. □

Note. I thought this lecture was very cute! It feels nice to go back to interesting geometric ideas, after struggling with tricky formalisms like immersions, submersions, and the general definition of transversality for a few lectures.

6 February 10th, 2021

Today we'll continue discussing homotopy and actually generalize it slightly, to prove facts about stability. We also introduce Sard's theorem. There will be no class next Tuesday due to a university holiday (President's Day).

6.1 More on Stability

Recall that the definition of homotopy involved a smooth deformation of two maps $X \to Y$ between manifolds, parameterized by a varying weight parameter in $[0, 1]$. However, the parameter doesn't necessarily always have to be in $[0, 1]$. We can generalize it to arbitrary parameter spaces.

Definition 6.1 (Family of maps)**.** Consider a subset $B \subset \mathbb{R}^n$, and two manifolds X and Y. A *family of maps* $\{F_b : X \to Y\}_{b \in B}$ parameterized by set B is defined to be simply a smooth map $F : B \times X \to Y$, where we let $F_b(x) = F(b, x)$ for all b.

Note that the above definition is still valid even when $B \subset \mathbb{R}^n$ is not a manifold, as we have a working definition for smoothness of functions on a subset of the reals by extending to an open neighborhood. We won't be considering any pathological examples of parameter spaces here, but notice that if we let $B = [0, 1] \subset \mathbb{R}$, we recover the definition of homotopy.

Here's a question that's quite interesting. Given two manifolds X and Y, can we give the set of maps $\{f : X \to Y\}$ the structure of a topological space, or perhaps even a manifold? The answer to the first question is **yes**, as there is a topology on the set of all maps. However, it turns out that this topology of smooth maps is unfortunately not a manifold, as it is simply too large.

Now, recall the notion of stability from Definition 5.7, which describes properties that are invariant under small deformations. We can restate this equivalently for families of maps parameterized by B, rather than $[0, 1]$, by asserting that the property holds for an open neighborhood of the origin in B, rather than an interval $[0, \epsilon) \subset [0, 1]$. In particular, it turns out that to prove Proposition 5.11 for the case of submersions, we simply need to make sure that the set of $k \times \ell$ matrices of full rank (in this case, the Jacobian) is an open subset of $\mathbb{R}^{k \times \ell}$. We can do this in a number of ways, such as by checking that the determinant of any $\ell \times \ell$ minor is nonzero.

To finish the proof, we have by this openness property that for each point $(0, x)$ within the product space $B \times X$, there is an open rectangle around that point where a function f remains a submersion. Then, we can use the compactness of X to find a finite subcover of this open cover. The intersection of the projections of all of these rectangles (which are finite in number) onto B is open, so this gives us an open neighborhood of B where the function $F(b)$ remains a submersion on all of X, as desired.

6.2 Sard's Theorem

Let's review a couple of definitions. Recall that given a smooth map $f : X \to Y$, we say that $y \in Y$ is a regular value of f when $f \pitchfork \{y\}$. Similarly, we have regular points in X. The complement of this set is the critical points in X, and their image forms the critical values in Y.

Now imagine that we consider the case when $\dim Y \geq \dim X$, and we consider the fibers $f^{-1}(y)$ for each $y \in Y$. When y is a regular value, the preimage is a manifold by Proposition 3.18. However, at critical values, the fibers become ill-behaved and potentially cause catastrophic changes in the shape of the manifold at regular values close to them (see Example 4.4). The problem with this picture, however, is that we may not know if a function has many regular values at all! This is what motivates Sard's theorem, which states that **there are not too many critical values**.

Proposition 6.2 (Sard's theorem)**.** *If $f : X \to Y$ is a smooth map, then the set of critical values is a set of measure zero*[7] *in Y. In particular, it cannot contain any open subset of Y.*

An important and immediate corollary is the following.

Corollary 6.2.1. *The locus of regular values is dense in Y.*

This means that in effect, any fiber of a value $y \in Y$ is either a manifold, or arbitrarily close to a manifold, in the sense that it becomes a manifold if y is perturbed arbitrarily slightly.

[7] This is not an analysis course, so we won't dwell too much on the intricate definition of measure (Borel, Lebesgue). However, you can think of measure zero in the following sense. A set $A \subset \mathbb{R}^n$ has *measure zero* if for any $\epsilon > 0$, there exists a collection of cuboids $R_i \subset \mathbb{R}^n$ covering that set with total volume less than ϵ. We can extend this to manifolds by asserting that this property holds for any coordinate chart.

7 February 17th, 2021

Today we start moving into the second part of the book [GP10], beginning with Chapter 9. Our general goal will be to understand the geometry of maps $f : X \to Y$ between manifolds, and in particular, the local picture (as this is much simpler than the global picture).

7.1 Non-Degenerate Critical Points

There are generally two cases of maps between manifolds: $\dim X \leq \dim Y$ and $\dim X \geq \dim Y$. We'll focus today on the second case, $\dim X \geq \dim Y$. When $\mathrm{d}f_x$ is surjective, we know that f is a submersion at x and locally looks like a projection map between vector spaces (Proposition 3.15). However, what about the case when $\mathrm{d}f_x$ is not surjective? In other words, what does the geometry of f look like at *critical points*?

Example 7.1 (Non-isolated critical values)**.** Consider a sheet of paper $P \subset \mathbb{R}^3$ lying on a flat table. If we were to bend the paper vertically into a "U" shape and project onto a vertical plane orthogonal to that cross-section, then the resulting map $f : P \to \mathbb{R}^2$ has critical values on the entire x-axis. This still has measure zero, but is not an isolated set (in particular, it has dimension 1).

In general we can't say much about the behavior of critical values, except Sard's theorem. So for now, we will specialize to the simplest case where $Y = \mathbb{R}$ is one-dimensional, and we consider the most basic critical points of the map, which we call *non-degenerate*.

Definition 7.2 (Non-degenerate critical point)**.** Consider a function $f : \mathbb{R}^n \to \mathbb{R}$, which has critical points precisely at those $x = (x_1, \ldots, x_n)$ where $\frac{\partial f}{\partial x_i} = 0$ for all i. We say that x is a *non-degenerate critical point* of f if the determinant of the Hessian matrix $H_{ij} = \partial^2 f / \partial x_i \partial x_j$ is nonzero.

The way to think about non-degenerate critical points is in terms of Taylor's theorem. Because the linear term is zero and the quadratic term is nondegenerate,[8] the local behavior of the function f near critical point x looks something like the Taylor expansion

$$f(x + v) \approx f(x) + \frac{1}{2} v^T H v + O(|v|^3).$$

This intuition is pretty much spot on to motivate Morse's lemma on quadratic forms.

Proposition 7.3 (Morse's lemma)**.** *If p is a non-degenerate critical point of f, then there exist local coordinates $x_1, \ldots, x_{k+\ell}$ on an open neighborhood of p such that*

$$f(x_1, \ldots, x_k) = f(p) + x_1^2 + x_2^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_{k+\ell}^2.$$

Then, we say that p is a critical point of signature (k, ℓ), which is the same as the signature of the quadratic form described locally by the Hessian matrix of f at p.

As an easy-to-visualize example, let's consider $\mathbb{R}^2$ for a moment.

[8] What does it really mean for the determinant of the Hessian to be nonzero? Why can't we just use the weaker condition that the Hessian matrix itself is nonzero? Well, we can view the Hessian as a kind of quadratic form on local coordinates near x. By the spectral theorem, there is a way to diagonalize H into a sum-of-squares of orthogonal components (principal axes), weighted by their respective eigenvalues. The determinant is nonzero when none of these eigenvalues are zero. Non-degenerate symmetric bilinear forms on $\mathbb{R}^n$ can be classified as having signature (k, ℓ) referring to the number of positive and negative eigenvalues, where $k + \ell = n$.

Example 7.4. In $\mathbb{R}^2$, then there are three possible signatures for non-degenerate critical points. If the signature is $(2,0)$, then f has a local minimum. If $(0,2)$, then it is a local maximum. Otherwise, if the signature is $(1,1)$, then we have a saddle point.

Joe mentions as an aside that in the doughnut example from Example 4.4, the way that fibers of the projection map $T \to \mathbb{R}$ change at critical points is related to their Hessian signatures, which are $(2,0)$, $(1,1)$, $(1,1)$, $(0,2)$ respectively. This is the subject of *Morse theory.*

7.2 Morse Functions

In this interlude, we'll discuss some juicy details about Morse functions, which are really nice!

Definition 7.5 (Morse function)**.** A smooth map f is called a *Morse function* if all critical points are non-degenerate.

Proposition 7.6. *If X is compact and $f : X \to Y$ is a Morse function, then f has only finitely many critical points.*

Proof. Use Morse's lemma to show that each critical point is isolated, as within an open neighborhood of a non-degenerate critical point, the derivative of f is nonzero except at that point. Then, by compactness, any set of isolated points must be finite. □

There is a key fact, related to Sard's theorem, that Morse functions are essentially *ubiquitous.* In other words, there are some measure-theoretic settings where we can say that "almost all" functions between two manifolds are Morse functions.

Example 7.7. If $X \subset \mathbb{R}^n$, then in the dual space consisting of linear functionals $\ell : \mathbb{R}^n \to \mathbb{R}$, we can construct an isomorphism $(\mathbb{R}^n)^* \cong \mathbb{R}^n$ using the standard inner product on reals. This lends a measure to the space of linear functionals. Within this space, it can be shown the set of non-Morse functions on X has measure zero.

7.3 The Tangent Bundle

Finally, Joe gives us a bit of a teaser for some of the topics for next week. Recall that many definitions in mathematics have changed throughout the years, and geometry is no exception. So far we have been using the 19th century definition of manifolds as subsets embedded in $\mathbb{R}^n$, but there is a more abstract modern definition that has superseded the need for an embedding.

Example 7.8. Consider $\mathbb{RP}^2$, the real projective plane, which can be viewed as the quotient a sphere S^2 under the equivalence relation identifying a point with its reflection about the origin. Even though this clearly a manifold, it is unclear how one would embed it in $\mathbb{R}^n$.

Let's give one more example of this. Recall that at each point in a manifold $x \in X^k$, there is a tangent space $T_x(X)$, which is a vector space of dimension k. What if we were to somehow take a disjoint union of all the tangent spaces at every point, to form a manifold of dimension $2k$? This is a fundamental object in differential topology and has a cool name.

Definition 7.9 (Tangent bundle)**.** The *tangent bundle* of a manifold $X \subset \mathbb{R}^n$ is a subset of $X \times \mathbb{R}^n$ given by the locus

$$TX = \{(x, v) : v \in T_x X\}.$$

It can be shown that this is a submanifold of $X \times \mathbb{R}^n$ with dimension $2n$, and the fibers of the projection map $(x, v) \mapsto x$, which is a submersion, are precisely the tangent spaces of X.

The tangent bundle is also a natural object in the modern definition of manifolds.

8 February 22nd, 2021

Today we discuss three topics. First, we summarize the book on embeddings of manifolds. Next, we cover various modern abstract definitions of what a (differentiable) manifold is, which is not covered in the textbook. Finally, if time permits, we introduce the notion of a manifold with boundary.

8.1 Embeddings of Manifolds

Suppose that we have a k-manifold $X \subset \mathbb{R}^N$. There are two statements about how we might be able to embed a given manifold in a higher-dimensional Euclidean space, and one of them is better than the other.

Proposition 8.1. *A compact k-manifold X can be embedded in $\mathbb{R}^{2k+1}$.*

Proof. Our method of attack will be to iteratively apply certain *projection* maps $\mathbb{R}^N \twoheadrightarrow \mathbb{R}^{N-1}$ to the Euclidean space, each time reducing the dimension by 1. In particular, given some unit vector $v \in \mathbb{R}^N$, we have a map $\pi_v : \mathbb{R}^N \to \langle v \rangle^{\perp} \cong \mathbb{R}^{N-1}$ defined by

$$\pi_v(w) = w - \langle w, v \rangle \cdot v.$$

Now the question we ask is, when is the map $\pi_v|_X$ an embedding? Clearly this map is proper because X is compact. Then, using the definition of an embedding, there are two possible failure modes: the map is not injective, or the map is not an immersion. It suffices to show that the map is both injective and an immersion whenever $N > 2k + 1$.

For the first part, we will show injectivity by a dimension counting argument. Consider the map $X \times X \setminus \Delta \to S^{N-1}$ given by $(u, w) \mapsto \frac{u-w}{\|u-w\|}$, where Δ is the diagonal. Note that v is not in the image of this map if and only if π_v is injective on X. However, when $N - 1 > 2k$, we know by dimension counting that the image of this map is a subset of measure zero in S^{N-1}. Therefore, we conclude that for almost all $v \in S^{N-1}$, the projection map π_v is injective.

For the second part, how can we check that $\pi_v|_X$ is an immersion? The key idea is to consider the tangent bundle $TX \subset \mathbb{R}^{2N}$, with dimension $2k$. We have a submanifold $TX_0 = \{(u, 0) : u \in X\}$. Then, we can once again define a map $TX \setminus TX_0 \to S^{N-1}$ taking $(u, w) \mapsto \frac{w}{\|w\|}$, and we conclude that the set of v for which π_v is not an immersion has measure zero.

Therefore, there exists some projection map π_v such that $\pi_v|_X$ is an embedding, and we finish the proof by applying a straightforward induction on N. □

As alluded to above, there is a better theorem that reduces the required embedding dimension by 1. In fact, you can construct examples where this second bound is tight. However, it is somewhat more difficult to justify, so we will omit the proof in this lecture.

Proposition 8.2 (Whitney)**.** *A compact k-manifold X can be embedded in $\mathbb{R}^{2k}$.*

8.2 The Modern Definition of a Manifold

According to [GP10], a manifold is defined to be a certain subset of $\mathbb{R}^N$. However, the modern definition of a manifold (since c. 1930) is as an *abstract object*, a set with a given structure, rather than as a subset of an ambient real vector space. This is not in the course syllabus or the textbook, but Joe believes that it's an important topic to contextualize some key ideas in geometry.

To start out, there are generally two *different* definitions of objects that are called manifolds: topological and smooth ($\mathcal{C}^\infty$). The former is much simpler than the latter, as it just requires a single point-set condition on the structure of the space.

Definition 8.3 (Topological manifold)**.** A topological space X is called a *topological manifold* of dimension k if it is Hausdorff, second countable, and for all $x \in X$, there exists a neighborhood $U \subset X$ of x homeomorphic to $\mathbb{R}^k$.

Note. Why must we require that X has nice properties in this definition? As a counterexample, take the real line with two origins, which is a non-Hausdorff topological space constructed by taking two copies of $\mathbb{R}$ and associating all points between these copies, except 0. This satisfies the homeomorphism property, but we probably don't want to consider this to be a manifold, as it's very weird. The second countable hypothesis will be necessary to embed a manifold in $\mathbb{R}^N$.

Now we can talk about smooth manifolds, which have $\mathcal{C}^\infty$ structure. There are three definitions. The first is from the textbook, which we've covered before in Definition 1.5. However, as the authors note, this definition has been considered "out of date" since at least the 1950s. The next two definitions are given below.

Definition 8.4 (Smooth manifold, second definition)**.** A topological manifold X is called *smooth* if there exists an *atlas* consisting of coordinate charts, such that for all pairs of coordinate charts $\phi : U \to \mathbb{R}^k$ and $\psi : V \to \mathbb{R}^k$ such that $U \cap V \neq \emptyset$, the transition map $\phi \circ \psi^{-1} : \mathbb{R}^k \to \mathbb{R}^k$ is a diffeomorphism of open sets in $\mathbb{R}^k$.

We want to be a bit careful with the above definition though, as there are multiple possible ways to specify a coordinate chart or atlas. So, we need to actually amend the second definition by specifying an equivalence class of atlases. Here's an equivalent third definition.

Definition 8.5 (Smooth manifolds, third definition)**.** If X is a topological space, and U is an open subset of X, then let $\mathcal{C}(U)$ be the set of continuous forms on U. When X is a topological manifold and U has a coordinate chart $\phi : U \to \mathbb{R}^k$, then $\mathcal{C}(U)$ forms a algebraic ring structure. This ring has a subring $\mathcal{C}^\infty(U)$, which consists of the forms f on U such that $f \circ \phi^{-1}$ is smooth. We call X a *differentiable manifold* if this subring $\mathcal{C}^\infty(U)$ is not dependent on the choice of ϕ.

Note. This idea of associating every open set in a manifold a local algebraic structure is called a *sheaf*, in algebraic topology. We can say that a topological manifold is smooth if its sheaf of continuous functions, $\mathcal{C}(X)$, has a subsheaf $\mathcal{C}^\infty(X)$.

It's clear from these constructions that $\mathcal{C}^\infty$ manifolds are a subset of topological manifolds, and likewise, some of the original examples of smooth manifolds were spheres (e.g., S^7). However, there is more than one distinct smooth manifold, up to diffeomorphism, that is *topologically* homeomorphic to a given smooth manifold like S^7.

Exercise 8.1. How might we adapt the definition of tangent space to the second or third definitions of a smooth manifold? There is no longer an enclosing real vector space, so we cannot simply let the tangent space be a linear subspace of $\mathbb{R}^N$. It turns out that there is a very natural definition in both of these cases, in terms of *derivations.*

Exercise 8.2. How do we know that a manifold X in these definitions is embeddable in a real vector space $\mathbb{R}^N$, i.e., showing that these modern definitions are equivalent to our main one? This involves using the second countable axiom.

8.3 Manifolds with Boundary

We'll introduce the basic idea of manifolds with boundary here, and leave the bulk of the discussion for Wednesday's lecture. First, consider some examples.

Example 8.6. The following objects are all not manifolds due to their closed boundary. The closed unit interval $[0,1] \subset \mathbb{R}$, the closed n-disk $D^n = \overline{S^n}$, and the closed upper half-plane $H^n \subset \mathbb{R}^n$ given by $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

It turns out that these definitions are actually representative of all manifolds without boundary. Instead of requiring all points to look locally like the reals, we instead allow them to look like subsets of H^n as well, in which case they are "on the boundary."

Definition 8.7 (Manifold with boundary)**.** We say that $X \subset \mathbb{R}^N$ is a *manifold with boundary* of dimension k if for all $x \in X$, there exists an open neighborhood $U \subset X$ of x that is diffeomorphic to an open subset of H^k.

The key fact about manifolds with boundary is that if an open neighborhood has a coordinate chart to H^k that contains its boundary, then there is no other coordinate chart that goes to H^k but does not contain the boundary. In other words, any diffeomorphism of H^k to itself maps the boundary to the boundary.

9 February 24th, 2021

Today we discuss manifolds with boundary and prove some of the classic results involving them, covering the first two chapters in Section 2 of [GP10]. Joe Harris began the lecture with a 25-minute interlude defining presheaves, sheaves and categories over $\mathcal{C}^\infty$ manifolds, but this went over my head, so I didn't take notes on it.

9.1 Vector Bundles

Recall Definition 7.9 gives us a way of unifying all the tangent spaces to a k-manifold at each point, which produces a manifold of dimension $2k$. Notice that the tangent bundle TX satisfies the criterion that the projection map $\pi_1 : (x, v) \mapsto x$ is a submersion from a $2k$-manifold to a k-manifold, with fibers of dimension k diffeomorphic to the tangent spaces T_xX. Here we introduce a slightly more general notion.

Definition 9.1 (Vector bundle)**.** A *vector bundle* $\pi : E \to B$ is a submersion of manifolds whose fibers $E_b = \pi^{-1}(b)$ have the structure of real vector spaces, and the vector addition and scalar multiplication maps

$$+ : E \times_B E \to E,$$
$$\times : \mathbb{R} \times E \to E.$$

are both smooth.[9] Here, E is called the *total space* of the bundle, or sometimes, E is just called the vector bundle itself.

The tangent bundle TX is a nontrivial special case of the vector bundle whose fibers are T_xX, and it can be constructed for any smooth manifold X. Joe asks us to verify as an exercise that the tangent bundle satisfies our two smoothness properties.

Example 9.2. The infinite Möbius strip M, constructed by taking $[0, 1] \times \mathbb{R} \big/ [(0, t) \sim (1, -t)]$, is a vector bundle of 1-dimensional vector spaces over the circle S^1. The cylinder $\mathbb{R} \times S^1$ is also a vector bundle over S^1, which is called the *trivial bundle*.

Vector bundles are a fundamental construct in differential topology, and we will see this type of object be used again later in this course.

9.2 Manifolds with Boundary (cont.)

Last lecture, we defined what it means for a subset $X \subset \mathbb{R}^N$ to be a manifold with boundary. Now, we will fill in holes in understanding by proving some useful facts about manifolds with boundary, which essentially are analogous to standard facts about manifolds.

Proposition 9.3. *The closed half-plane H^k is not diffeomorphic to $\mathbb{R}^k$.*

Proof. One approach is to use the inverse function theorem at the point $(0, 0)$ in H^k, which immediately kills the problem. However, we can also prove a stronger fact that H^k is not *homeomorphic*

[9] In this expression, the notation $E \times_B E$ is called a *fiber product*, which has the structure of a manifold when the projection map $\pi : E \to B$ is a submersion. In this case, it consists of all pairs (x, v_1) and (x, v_2) of a single point and two tangent vectors at that point.

to $\mathbb{R}^k$ using pure algebraic topology and avoid differential notions. Suppose for the sake of contradiction that $\phi : H^k \to \mathbb{R}^k$. Then, suppose that we removed $0 \in H^k$ and $\phi(0) \in \mathbb{R}^k$. It follows that ϕ restricted to a subset of its domain is still a homeomorphism between $H^k \setminus \{0\}$ and $\mathbb{R}^k \setminus \{\phi(0)\}$.

However, the first space $H^k \setminus \{0\}$ is homotopy equivalent to a point (contractible), so its homotopy groups are all trivial. Meanwhile, the second space $\mathbb{R}^k \setminus \{\phi(0)\}$ is homotopy equivalent to S^{k-1}, so its $(k-1)$-th order homotopy group is $\pi_{k-1}(\mathbb{R}^k \setminus \{\phi(0)\}) = \mathbb{Z}$. However, homotopy groups are supposed to be invariant up to isomorphism between simply-connected homeomorphic topological spaces, so we have a contradiction. □

This statement essentially guarantees that any point in a manifold with boundary must either map to a boundary point or an interior point of H^k under a coordinate chart, but it cannot map to both kinds of points under different coordinate charts. This is because if a point $x \in X$ has a neighborhood $U \subset X$ and two coordinate charts ϕ and ψ which map x to a boundary point and an interior point respectively, then $\phi \circ \psi^{-1}$ would be a local diffeomorphism between an open ball and a semi-closed half-disk in $\mathbb{R}^k$. This contradicts Proposition 9.3.

Definition 9.4 (Boundary point)**.** If X is a manifold with boundary, then we call $x \in X$ a *boundary point* of X if its image under any diffeomorphism of a neighborhood of x to an open subset of H^k lies on the boundary.

We'll see how to apply these results to 1-manifolds with boundary during our next lecture, and perhaps also cover a generalization of the Borsuk-Ulam theorem.

10 March 3rd, 2021

Today we talk about properties of manifolds with boundary, classify manifolds with boundary of dimension 1, and then use this result to prove a classic result in topology.

10.1 Manifolds with Boundary (cont.)

Recall that Definition 8.7 gives us a way to talk about manifolds that have boundaries. A k-manifold with boundary M has a *boundary* ∂M, which is a submanifold without boundary of dimension $k-1$ consisting of all points that map to ∂H^k under a local diffeomorphism of M to H^k.

Note. The word *boundary* in the case of manifolds is different from the same term when used in point-set topology, which refers to the closure minus the interior of a set of points in a topological space. Here, boundary is an intrinsic property of manifolds, rather than an extrinsic property of sets of points viewed in a topological space.

Now we'll state some basic facts about manifolds with boundary.

Proposition 10.1. *If X is a manifold and Y is a manifold with boundary, then $X \times Y$ is a manifold with boundary, and $\partial(X \times Y) = X \times \partial Y$.*

Note that this is not true if both X and Y are manifolds with boundary. The classic counterexample is $[0,1] \times [0,1] = [0,1]^2$, which is not a smooth manifold with boundary due to the corner points. (However, you can note that it is still a *topological* manifold with boundary.)

Definition 10.2 (Boundary derivative)**.** If $f : X \to Y$ is a map between manifolds with boundary, then let $\partial f : \partial X \to Y$ be the restriction of f to domain $\partial X \subset X$. At any boundary point $x \in \partial X$, there exist two tangent spaces $T_x(\partial X) \subset T_x(X)$, corresponding to the domains of $\mathrm{d}(\partial f)_x$ and $\mathrm{d}f_x$, respectively. These have dimension $\dim X - 1$ and $\dim X$.

Definition 10.3 (Transversality with boundary)**.** If X is a manifold with boundary, Y is a manifold with submanifold $Z \subset Y$, and $f : X \to Y$ is a smooth map, then $f \pitchfork Z$ if and only if both $\partial f \pitchfork Z$ and $f|_{\mathrm{Int}(X)} \pitchfork Z$. In other words, we have two conditions:

- For all $x \in f^{-1}(Z)$,
$$\operatorname{im} \mathrm{d}f_x + T_{f(x)}Z = T_{f(x)}Y.$$
- For all $x \in f^{-1}(Z) \cap \partial X$,
$$\operatorname{im} \mathrm{d}(\partial f)_x + T_{f(x)}Z = T_{f(x)}Y.$$

To visualize the previous definition, we can once again think about the special case when f is an embedding. Then, transversality is essentially the same when the boundary of $f(X)$ does not intersect with Z. However, in the places where the boundary ∂X intersects with Z, we must also have that the boundary itself is transversal (and not tangent) at those points.

Proposition 10.4. *If $f \pitchfork Z$, where $f : X \to Y$ is a map from a manifold with boundary X to a manifold Y, then $f^{-1}(Z)$ is a manifold with boundary, and $\partial f^{-1}(Z) = \partial X \cap f^{-1}(Z)$.*

Example 10.5. Consider a line, which is a 1-manifold, intersecting with a closed disk, which is a 2-manifold with boundary. If the line is tangent, then the intersection is not transversal. Otherwise, the line intersects the disk to form a closed line segment, which is a 1-manifold *with* boundary.

We can also define regular values for manifolds with boundary by requiring that the restricted map ∂f is a submersion from ∂X to Y. As before, the key idea is to add additional conditions to our definition in order to take care of the boundary as a special case. Sard's theorem (Proposition 6.2) also holds for manifolds with boundary.

10.2 Brouwer's Fixed Point Theorem

Let's talk about Brouwer's fixed point theorem and its corollaries. We're going to prove this theorem using the tools of differential topology, instead the more classical approach using algebraic topology. First, we need to classify all 1-manifolds with boundary, for reasons that will be apparent later.

Proposition 10.6 (Classification of 1-manifolds). *Any compact, connected, 1-manifold is diffeomorphic to S^1. Furthermore, any compact, connected, 1-manifold with boundary is diffeomorphic to either S^1 to $[0, 1]$.*

Proof. Start at a point; a neighborhood of that point is diffeomorphic to either $(0, 1)$ or $[0, 1)$. You can then walk along the curve and continue until you either stop, which gives you $[0, 1]$, or hit the point that you started at, which gives S^1. A full proof is given in the appendix of [GP10]. □

Corollary 10.6.1. *If X is a compact 1-manifold with boundary, then ∂X is a set containing a finite and even number of isolated points.*

This corollary is already fairly striking (wow, parity!), but it lets us prove a very useful theorem about *retractions* in a purely differential way.

Proposition 10.7 (Retractions onto the boundary do not exist). *If X is a compact manifold with boundary in any number of dimensions, then there does not exist a smooth map $g : X \to \partial X$ such that $g|_{\partial X}$ is the identity map on ∂X.*

Proof. Assume the opposite, for the sake of contradiction. Choose any point $p \in \partial X$ that is a regular value of g. Then, by Proposition 10.4, $g^{-1}(p)$ is a 1-manifold with boundary, and it boundary is given by $\partial(g^{-1}(p)) = g^{-1}(p) \cap \partial X \subset \partial X$. However, by Corollary 10.6.1, the number of points in $\partial(g^{-1}(p))$ is even, so in particular, there must be some other point $q \neq p$ in ∂X such that $g(q) = p$. This contradicts the assumption that $g|_{\partial X}$ is the identity, so we conclude. □

Now let's see a "highly amusing" application of this retraction theorem.

Corollary 10.7.1 (Brouwer's fixed point theorem). *Any smooth map $f : D^n \to D^n$ from the closed unit disk to itself has a fixed point.*

Proof. Assume the opposite, that there exists some f without a fixed point. Then we can define a map $g : D^n \to \partial D^n \cong S^{n-1}$ drawing a ray from $f(x)$ to x and marking the first intersection of that ray with ∂D^n. This is a smooth map and a retraction, since you can write it down explicitly by using some quadratic formula nonsense, so we contradict Proposition 10.7. □

11 March 8th, 2021

Previously, we talked about transversality, or what it means for two manifolds to intersect in a "general position" that doesn't involve tangency. Now we would like to consider the specific case when two manifolds intersect at isolated points, and we analyze what happens when one of these manifolds is smoothly deformed.

11.1 Intersection Theory

Of course, when two manifolds intersect transversely, the intersections are isotopic under a smooth transformation, as transversality is a stable property. The non-transverse case is much more interesting and brings rise to the topic of *intersection theory.* We have a choice between two options:

- (Chapter 2) Count the number of intersection points modulo 2, or
- (Chapter 3) Introduce *orientation* on manifolds and *intersection indices.*

In both cases, we will achieve our goal of obtaining some integer invariant that is the same no matter how the manifold is deformed. However, just to get started with setting up the picture, we will only cover the modulo 2 case for now and leave orientation for a future lecture.

Definition 11.1 (Complementary dimension)**.** Suppose that Y is a manifold of dimension n, and $X, Z \subset Y$ are submanifolds of dimension ℓ and k. We say that X and Z are of *complementary dimension* if $k + \ell = n$. This allows X and Z to intersect transversely at an isolated point.

Our general setup will be to have compact submanifolds $X, Z \subset Y$, and we will try to *count* the value $\#(X \cap Z)$ as X and Z smoothly deform. More generally, we can replace the X with an arbitrary function $f : X \to Y$, and then we will want to instead count $\#f^{-1}(Z)$.

Proposition 11.2 (Moving lemma)**.** *Given a manifold Y of dimension n, a submanifold $Z \subset Y$, and a smooth function $f : X \to Y$ of complementary dimension,*[10] *there exists a deformation f_s of f that is transverse to Z. To formalize this, the moving lemma has two parts:*

1. *There exists a connected manifold S and family of maps $F : S \times X \to Y$ such that for almost all $s \in S$, $f_s \pitchfork Z$.*
2. *Choose any $s \in S$ such that $f_s \pitchfork Z$. Then, $\#(f_s^{-1}(Z))$ modulo 2 is independent of the choice of s. We can notate this using as an intersection number $I_2(f, Z) \in \mathbb{Z}/2\mathbb{Z}$*

Proof. We first tackle part 1 of the lemma. Given $f : X \to Y$, we want to deform f. Notice that if $Y = \mathbb{R}^n$, we could just take $f_s = f + tg$ for some smooth function $g : X \to \mathbb{R}^n$.

To construct deformations of f, our trick will be to construct many small deformations in local neighborhoods of f, with bump functions then add those up to form a global deformation. Let $h : \mathbb{R}^k \to [0, 1]$ be a bump function that is identically equal to 0 outside a ball of radius 1 at the origin, but is equal to 1 inside a ball of radius $\frac{1}{2}$. For $p \in X$ and $q = f(p) \in Y$, suppose that we have coordinate charts of neighborhoods $p \in U$ to $B_1(0)$ and $q \in V$ to $\mathbb{R}^n$. Then, for $s \in \mathbb{R}^n$, we define $f_s : X \to Y$ using these coordinate charts implicitly so that

$$f_s(x) = \begin{cases} f(x), & x \notin U, \\ f(x) + h(x) \cdot s, & x \in U. \end{cases}$$

[10]Here, we will also assume that X is compact. However, Gaurav notes that you can use a partition of unity to reduce the general case to the compact case, so compactness is not required in the statement in [GP10].

Using this method, we obtain by construction a family of maps $F : \mathbb{R}^n \times X \to Y$ that is a submersion in U. This means that $f_s|_U$ must intersect transversally with Z for almost all s.

We can extend this construction by choosing a finite atlas of m coordinate charts for open sets covering X, assuming that X is compact. Then, we analogously get a family of maps $F : (\mathbb{R}^n)^m \times X \to Y$ that is a *global* submersion, and we conclude that $f_s \pitchfork Z$ for almost all s.

The second part of this lemma follows from the fact that the boundary of a compact 1-manifold has an even number of points, as stated in Corollary 10.6.1. Essentially, assume that we have two functions $f_s \pitchfork Z$ and $f_{s'} \pitchfork Z$. Then, part (a) tells us that there exists a homotopy $F : I \times X \to Y$ such that $F(0, x) = f_s(x)$, $F(1, x) = f_{s'}(x)$, and $F \pitchfork Z$. By transversality, $F^{-1}(Z)$ is a compact 1-manifold with boundary. By Proposition 10.4,

$$\partial F^{-1}(Z) = F^{-1}(Z) \cap (\{0, 1\} \times X) = f_s^{-1}(Z) \coprod f_{s'}^{-1}(Z).$$

Therefore, since the boundary of a compact 1-manifold has even cardinality, the total number of points in $f_s^{-1}(Z)$ and $f_{s'}^{-1}(Z)$ is even, so they must have the same parity. □

11.2 Examples of Intersections

We'd like to present a couple small examples of intersection theory in practice, trying to make the moving lemma more concrete. In each case, Y is a 2-manifold surface, and we will consider submanifolds X and Z, which are both curves.

Example 11.3. Consider the maximally non-transverse case when $X = Z$ as circles S^1 embedded in a cylinder. Then, as X is perturbed slightly by shifting it upward, the number of intersections becomes 0. However, when X is kind of rotated about a horizontal axis, the number of intersections is 2 almost everywhere. In either case, the parity of $I(X, Z)$ is even.

Example 11.4. Consider the same example where Y is a Möbius strip, and $X, Z \subset Y$ are identical embeddings of S^1 on the mid-line of the Möbius strip. If we shift X slightly, then we can obtain just a *single* intersection with Z. Therefore, by the second part of the moving lemma, $I(X, Z)$ must always have odd parity. Qualitatively, this shows that we cannot find a deformation of the mid-line X of the Möbius strip that does not intersect with X.

12 March 10th, 2021

In today's lecture, we continue discussing the modulo 2 intersection theory of manifolds.

12.1 Modulo 2 Intersection Theory

Let's start with a recap of modulo 2 intersection theory. Since this is our second time summarizing the topic, we will try to formalize and restate the hypotheses to make everything clear.

The basic setup is that we have a closed manifold[11] X, a submanifold $Z \subset Y$, as well as a function $f : X \to Y$, where $k = \dim X$, $\ell = \dim Z$, and $n = \dim Y = k + \ell$. Under these conditions, our key fact is Proposition 11.2, which tells us that we can find a family of deformations f_s similar to f such that almost all of them are transverse to Z. Furthermore, for any such parameter set s satisfying $f_s \pitchfork Z$, the number of intersection points $\#(f_s^{-1}(Z))$ modulo 2 is the same.

The proof of the first part of the lemma is trickiest. The key idea is to work locally and apply a *bump function* to perturb f, which is a $\mathcal{C}^\infty$ function that is 1 on a neighborhood of the origin but zero globally. We can construct such a function, for example, by carefully using $e^{-1/x}$. It's left as an exercise to adjust this to produce a desired bump function h such that $h(x) = 1$ within a ball of radius $\frac{1}{2}$, but $h(x) = 0$ outside a ball of radius 1.

Now, we can produce a map $F : \mathbb{R}^n \times X \to Y$ locally by $F(s, x) = f(x) + h(x) \cdot s$, going through a coordinate chart of X and Y. Since the derivative with respect to s is just 1, it follows that F is a submersion at all x within a ball of radius $\frac{1}{2}$. Then, by Sard's theorem, $f_s \pitchfork Z$ in a ball of radius $\frac{1}{2}$ for almost all $s \in \mathbb{R}^n$, since the critical values are a set of measure zero. Finally, we conclude by covering X with a finite collection of m open sets with coordinate charts (since X is compact), which induces a global family of maps $F : \mathbb{R}^{nm} \times X \to Y$.

In summary: for any map $f : X \to Y$ and submanifold $Z \subset Y$ where X is compact, we can always deform f to be transverse to Z, and the number of intersections modulo 2 will always be the same value in $\mathbb{Z}/2\mathbb{Z}$, independent of which homotopy we use to deform f. We denote this intersection number by $I_2(f, Z)$. Furthermore, if f is an embedding $X \hookrightarrow Z$, then we can also instead write $I_2(X, Z)$ between two submanifolds of Y.

Note that although the existence of a transverse deformation in part 1 of the moving lemma is very intricate and hard to prove, this is not actually how you construct such a deformation in practice. In particular, the moving lemma illustrates that *almost all* deformations in some sense will make the manifold intersect transversely. Therefore, in applications we can usually just take a very simple deformation and win immediately.

Example 12.1 (Self-intersection)**.** Consider $X = Z$, where X is a closed manifold of dimension n, and it is contained as a submanifold of Y, which has dimension $2n$. Then, the number $I_2(X, X)$ is well-defined under the moving lemma and is called the *self-intersection number* of X. The self-intersection number of the equator in S^2 is zero, implying that two loops on a sphere always intersect in an even number of points.

We saw that in Example 11.4 that the self-intersection of the mid-line of a Möbius strip is 1.

Example 12.2. More generally, for any $X, Z \subset S^n$ with dimensions $k, \ell < n$, we can apply a homotopy that sends X and Z to opposite hemispheres, so that they cannot intersect. Therefore, we must have that $I_2(X, Z) = 0$ always.

Example 12.3. Consider the real projective plane $Y = \mathbb{RP}^2$, which is the quotient of S^2 by the natural order-2 symmetry. Let $X = \mathbb{RP}^1$ sitting inside Y. We can also think of X and Y as the

[11] In differential topology, the word *closed* when describing a manifold means compact and without boundary.

sets of all 1-dimensional subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Then, $I_2(X, X)$ can be computed by deforming X to X' on a slightly different embedding of $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$. Therefore, $I_2(X, X) = 1$, since the only intersection is precisely the 1-dimensional subspace consisting of the intersection between these two 2-dimensional subspaces in $\mathbb{R}^3$.

Hopefully these examples illustrate that in practice, we can make such deformations whose existence is shown by Proposition 11.2 explicit.

12.2 The Boundary Theorem

Will prove one more fact that will lead to the beginnings of homology theory.

Proposition 12.4 (Boundary theorem)**.** *Suppose that we have a submanifold $Z \subset Y$ and smooth map $f : X \to Y$, such that $X = \partial W$ is the boundary of a manifold with boundary W. Furthermore, assume that $f : X \to Y$ extends to a map $g : W \to Y$. Then, $I_2(X, Z) = 0$.*

Proof. Deform Z so that $g \pitchfork Z$. Then $f^{-1}(Z) = \partial(g^{-1}(Z))$, and by Corollary 10.6.1, the cardinality of this set must be even. □

In other words, we have shown that the boundary of a manifold will always have zero intersection number with any other manifold. You can imagine this intuitively in the case of 1-dimensional Z as a closed curve alternating between being "inside W" and "outside W," where the total number of intersections must be even. This means that the boundary of a submanifold with one higher dimension is essentially trivial in intersection theory, which motivates homology.

13 March 15th, 2021

In today's lecture,[12] we continue mod 2 intersection numbers (section 2.4) and continuing to (section 2.5). We will finish chapter 2 on Wednesday and start on chapter 3 next week.

Recall that we have $Y \supset Z$, where Y has dimension n and Z has dimension $l < n$. Let X be a compact manifold of dimension k with map $f : X \to Y$. In the past lecture, we looked at when f is an embedding and assumed on $k + l = n$, and continued to use the Proposition 11.2.

13.1 Boundary Theorem, Revisited

A simple case of the boundary theorem is where W is some simply connected 2-manifold. Here, Z is some 1-dimensional curve. Then, the number of times Z enter W is equal to the number of times it leaves W. Hence, the number of intersection points $I_2(X, Z)$ is even.

When we work with intersection numbers in $\mathbb{Z}$, we will convert this to some kind of sign: points where Z enters W will have the opposite sign as points where Z leaves W. Not only is the number of intersection points even, we can match them!

13.2 Mod-2 Degree

Another special case is when $l = 0, k = n$. That is, $Z = \{y\}$ and X is also an n-dimensional manifold. Assume Y is connected. Notice that $I_2(f, \{y\})$ does not depend on the choice of y. If we have a different choice y', let W to be any path from y to y' to see $I_2(f, \{y\}) = I_2(f, \{y'\})$. The quantity $I_2(f, \{y\})$ is called the *mod-2 degree* $\deg_2(f)$.

We can state 2 theorems from last class in terms of $\deg_2(f)$:

1. Homotopic maps $X \to Y$ have same mod 2 degree,

2. If $X = \partial W$ and f extends to W, then $\deg_2(f) = 0$.

Example: Let $f : W \to \mathbb{R}^2$ be any smooth map, and suppose we know $f|_X$ where $X = \partial W$. Can we determine the values of f in the interior of W? The answer is no: from real analysis, we can add a bump function without changing $f|_X$ (with holomorphic function, this is true!).

However, for a given $p \in \mathbb{R}^2$, is p in the image of f? In this case, we have a sufficient condition.

WLOG let $p = 0$ and assume $0 \notin f(X)$ (otherwise the answer is yes). Then, we can define $\alpha_0 : X \to S^1, x \mapsto f(x)/||f(x)||$. If $f(w) \neq 0$ for all $w \in W$, then we can extend α_0 to a map $W \to S^1$ and therefore $\deg_2(\alpha_0) = 0$. Thus, if $\deg_2(\alpha_0) \neq 0$ then $0 \in f(W)$.

We arrive at half of the Fundamental Theorem of Algebra:

Proposition 13.1. *If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial with odd degree, then f has a zero.*

Proof. Consider $p_t(z) = tp(z) + (1-t)z^m$, which is a homotopy. Then, consider $p_t'(z) = p_t(z)/|p_t(z)|$, which are maps $\mathbb{C} \to S^1$. For a large enough circle around 0 we find that $p_t(z)$ is never 0, so $p_t'(z)$ is a homotopy. Hence, $\deg_2(p/|p|) = \deg_2(p_0/|p_0|)$. Since $p_0 = z^m$, $\deg_2(p_0/|p_0|) = m \mod 2$.

If m is odd, $\deg_2(p(z)/|p(z)|) = 1$, so 0 is in the image of f. □

[12] Today's lecture notes were contributed and typeset by Richard Xu.

13.3 Jordan-Brouwer Separation Theorem

Suppose we have C a circle in $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus C$ has 2 connected components: the interior and the exterior part. Is this true for *any* embedding $S^1 \to \mathbb{R}^2$, or any embedding $S^n \to \mathbb{R}^{n+1}$?

To answer this, we need to characterize points in the interior and exterior of $\mathbb{R}^2 \setminus f(S^1)$. Let $z \in \mathbb{R}^2$ not on $f(S^1)$, and let W be the interior of $f(S^1)$. Consider $\alpha_z : S^1 \to S^1, x \mapsto (x-z)/||x-z||$. If $\deg_2(\alpha_z)$ is odd, then α_z does not extend to all of W so $z \in W$.

In this case, $\deg_2(\alpha_z)$ is known as the *winding number*. We will show next class that if $\deg_2(\alpha_z)$ is even then z is in the exterior, which gives us a complete characterization.

How do we understand the winding number? Suppose we are at point z and is watching a car going around S^1 in binoculars. If we keep our binoculars focused on the car, then the winding number is the number of times we turn around when the car goes one around.

Additionally, choose $Z = \{v\}, v \in S^1$. Then, $\deg \alpha_z = \#\alpha_z^{-1}(v)$ for any regular value $v \in S$. This is equal to the number of intersections between S^1 and $\mathbb{R}_+ v$! In other words, we can draw a ray from z. If the ray intersects S^1 an odd number of times, we are in the interior. We now show that even degree implies z is in the exterior.

The full characterization is given by the theorem below:

Theorem 13.2 (Jordan-Brouwer separation)**.** *The complement of $X = f(S^1)$ in $\mathbb{R}^2$ has two connected components, the interior and exterior of X. The interior is compact with boundary X. Furthermore, the interior is the set of points z where* $\deg_2(\alpha_z) = 1$*, and the exterior is the set of points z where* $\deg_2(\alpha_z) = 0$.

For higher dimensions $f : S^n \to \mathbb{R}^{n+1}$ and $z \in \mathbb{R}^{n+1} \setminus S^n$, again define $\alpha_z : x \mapsto (x-z)/||x-z||$. Then, z is in the interior iff $\deg_2(\alpha_z)$ *is odd, which occurs iff a ray $\mathbb{R}_+ v$ coming from z intersects S^n in an odd number of points.*

We may not get to the full proof on Wednesday, so if you want to try the proof, there is a series of 11 problems in the book that can guide you through it!

14 March 17th, 2021

Today we discuss the Borsuk-Ulam theorem and related facts in topology. We also define what it means for a manifold to have *orientation*, which will be needed to refine our intersection theory.

14.1 The Borsuk-Ulam Theorem

To state and prove the version of Borsuk-Ulam in this lecture, we will need the notion of a winding number. Winding numbers were introduced last lecture, so let's review the definition.

Definition 14.1 (Winding number, modulo 2)**.** Suppose that X is a compact manifold of dimension n and $f : X \to \mathbb{R}^{n+1}$ is a smooth map, with point $z \in \mathbb{R}^{n+1} \setminus f(X)$. Then, we can define a map $\alpha_z : X \to S^n$ taking

$$\alpha_z(x) = \frac{f(x) - z}{\|f(x) - z\|}.$$

We define the *modulo 2 winding number* of f around z by $W_2(f, z) = \deg_2(\alpha_z)$.

Furthermore, whenever v is a regular value of α_z (which occurs almost everywhere by Sard's theorem), this is precisely equal to the cardinality of the set $f^{-1}(\mathbb{R}_+ v)$, or the number of times an infinite ray from z to v intersects f.

Proposition 14.2 (Borsuk-Ulam theorem)**.** *If $f : S^n \to \mathbb{R}^{n+1}$ is an odd function, meaning that for all $x \in S^n$, $f(-x) = -f(x)$, and 0 is not in the image of f, then $W_2(f, 0) = 1$. (This means, in particular, that f intersects every ray from the origin at least once.)*

Proof. A proof is given in the textbook; it is quite tricky and uses induction on n. □

The more classical statement of Borsuk-Ulam follows immediately as a corollary.

Corollary 14.2.1. *If $f_1, \ldots, f_n$ are smooth, odd functions from $S^k \to \mathbb{R}$, then $f_1, \ldots, f_n$ have a common zero.*

Proof. Define a map $f(x) = (f_1(x), \ldots, f_n(x), 0)$. If 0 is in the image of f, then we are done. Otherwise, by Proposition 14.2, we have $W_2(f, 0) = 1$, so there is some point $(0, \ldots, 0, \lambda) \in \operatorname{im} f$ for positive λ, which is a contradiction. □

Corollary 14.2.2 ("Meteorological theorem")**.** *If $g : S^n \to \mathbb{R}^n$ is a smooth function, then there exists a pair of antipodal points $z, -z \in S^n$ such that $g(z) = g(-z)$.*

Proof. Apply the previous result with $g(x) - g(-x)$, which is an odd function. □

The above corollary specifically when $n = 2$ is sometimes stated in popular culture as, "There are two points on opposite sides of the Earth with the same temperature and barometric pressure."

Corollary 14.2.3 (Ham sandwich theorem)**.** *Given n measurable sets in $\mathbb{R}^n$, it's possible to bisect them by an $(n-1)$-dimensional hyperplane so that the two sides have equal measure of all sets. (In particular, we can cut a ham sandwich in $\mathbb{R}^3$ so that the two halves have equal amounts of ham, cheese, and bread.)*

14.2 Orientation

The way we motivate this section is by trying to refine the sledgehammer of winding numbers modulo 2. Although I_2 gives us parity, there are often times when we would like to have a more specialized count of intersections. The key idea will be to define an intersection *index* for each point $p \in X \cap Z$, which is set to ± 1. We will define the index in such a way that the sum of all indices in the intersection is homotopy invariant.

Example 14.3. Consider the graph of $\sin x$ intersecting with the x-axis. Between every pair of adjacent intersections, the graph alternates from $\sin x$ being above to below the x-axis. We should assign these intersection points opposite indices, so that they cancel each other out.

With the goal in sight, we set our game plan. First we will need to define orientation for an individual vector space or manifold, then combine two orientations for $f : X \to Y$ and $Z \subseteq Y$, when $f \pitchfork Z$, to define the orientation of a preimage $f^{-1}(Z)$.

Definition 14.4 (Orientation)**.** Given two bases $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ for a vector space V, we say that (v_i) and (w_i) are *equivalently oriented* if the determinant of the change-of-basis matrix is positive. An *orientation* on a vector space is a choice of basis, belonging to one of these two equivalence classes (which we call positive/negatively oriented).

In $\mathbb{R}^n$, we usually specify the *standard orientation* to be given by the standard orthonormal basis $e_1, \ldots, e_n$. The order matters here, as swapping any two basis vectors would reverse the basis. Given an orientation for V (of dimension m) and an orientation for W (of dimension n), we can induce an orientation for $V \oplus W$ by simply concatenating their bases. Notice that this is not the same as $W \oplus V$, as it differs by sign $(-1)^{mn}$.

15 March 22nd, 2021

Today we continue our discussion of orientation, with the goal of extending intersection numbers from parities in $\mathbb{Z}/2\mathbb{Z}$ to general positive and negative integers.

15.1 Orientable Manifolds

Recall that an orientation of a vector space is just an assignment of each ordered basis to one of two signs, based on which equivalence class of $\mathrm{GL}_n(\mathbb{R})$ they lie in, with either positive or negative determinant. If a change-of-basis matrix has positive determinant, then the two bases it relates are equivalently oriented. Otherwise, they are oppositely oriented.

Observe that given a continuous family of bases on a vector space, every basis in this family must have the same orientation. This follows from the fact that $\mathrm{GL}_n(\mathbb{R})$ has two path-connected components based on sign of the determinant. Also, if the vectors in a basis are permuted, then the orientation changes by the *sign* of the permutation (even or odd number of inversions). This is because the determinant of a matrix representing an even permutation is $+1$, while the determinant of an odd permutation matrix is -1.

Definition 15.1 (Orientation-preserving)**.** If U and V are oriented vector spaces of the same dimension, we say that an isomorphism $f : U \to V$ is *orientation-preserving* if it takes the equivalence classes of corresponding orientations to each other.

We previously saw how we can define an orientation on $A \oplus B$ given orientations on A and B. As it turns out, we can generalize this to quotient vector spaces.

Exercise 15.1 (Quotient orientation)**.** Show that given a short exact sequence

$$0 \to A \xrightarrow{i} V \xrightarrow{\pi} B \to 0,$$

an orientation on any two of A, V, B determines an orientation on the third vector space.

So far, we've just been playing with symbols to define orientation on flat spaces. This starts to get much more exciting and useful when we introduce orientations on general manifolds.

Definition 15.2 (Orientable manifold)**.** A manifold X of dimension n is called *orientable* if there exists a *locally consistent* choice of orientation for $T_p(X)$ at each $p \in X$. This means that for each neighborhood $U \subset X$ of p with coordinate chart $\phi : U \to \mathbb{R}^n$, and point $q \in U$, the isomorphism of tangent spaces $T_p(X) \to T_q(X)$ induced by the differential $\mathrm{d}\phi_q^{-1} \circ \mathrm{d}\phi_p$ is orientation-preserving.

Many common manifolds are orientable, but some are not. The most celebrated example of a non-orientable manifold is the Möbius strip, where you can take a continuous path along the center line and end up with a basis for the manifold's tangent space with opposite orientation.

Note. We can also define a two-sheeting covering space of a connected manifold X consisting of pairs (p, o), where $p \in X$ is a point and o is an orientation of the tangent space $T_p(X)$. If this covering space has two disconnected components, then the manifold is orientable. Otherwise, there exists a path going from (p, o) to $(p, -o)$, so the manifold is non-orientable.

15.2 Orientation on a Boundary

Recall that a manifold with boundary X can be split up into its boundary ∂X, and its interior $X \setminus \partial X$. If $\dim X = n$, then $\dim \partial X = n-1$. It turns out that if $X \setminus \partial X$ is orientable, then ∂X is also orientable, so in this case, we simply write that X is orientable.

Proposition 15.3. *If a manifold X is orientable, then its boundary ∂X is orientable.*

Proof. At each point $x \in \partial X$, write $T_p(\partial X) \subset T_p(X)$ as a linear subspace of codimension 1. Then there canonically exists a short exact sequence

$$0 \to T_p(\partial X) \to T_p(X) \to N_p(X) \to 0,$$

where $N_p(X) = T_p(X)/T_p(\partial X)$ is the quotient vector space. Let $n \in N_p(X)$ be an outward-pointing vector, and use this as the definition of positive orientation on $N_p(X)$.[13] Then, we define an orientation on $T_p(\partial X)$ so that this short exact sequence commutes with orientation. □

Here's an important example of the above fact. Let X be a manifold (without boundary) of dimension n, and consider the direct product $I \times X$. This is a manifold with boundary

$$\partial(I \times X) = \{0,1\} \times X = X_0 \coprod X_1.$$

Using the standard orientation on I, we arrive at an orientation for $I \times X$. Then, applying Proposition 15.3, the orientation of $\partial(I \times X)$ consists of X_1, oriented equivalently with X; and X_0, oriented oppositely with X. We can abuse notation and write $\partial(I \times X) = X_1 - X_0$.

Definition 15.4 (Index)**.** Given a connected oriented 0-manifold, i.e., a single point, there are two possible orientations as always. However, there is only one basis, consisting of the empty set! Therefore, we give the orientation of this point by its *index*, which is a number ± 1.

Using indices, we can write a generalization of Corollary 10.6.1 using orientation.

Lemma 15.5. *The sum of indices on the boundary of a compact 1-manifold is zero.*

Here's one more puzzle to think about. Observe that $\mathbb{RP}^1 \cong S^1$ is orientable. However, $\mathbb{RP}^2$ is non-orientable, because removing a disk from the sphere S^2 before taking the quotient produces a Möbius strip as a subset of $\mathbb{RP}^2$.

Exercise 15.2. For each $n \geq 1$, is $\mathbb{RP}^n$ orientable?

[13] When X receives an inner product from the enclosing space $\mathbb{R}^N$, we can take *the* unique outward normal vector, but for the sake of just defining orientation, we pick any outward vector due to not having a canonical metric.

16 March 24th, 2021

Today we continue discussing orientation, with the goal of defining an oriented intersection number. This will allow us to generalize our modulo 2 intersection theory to oriented intersection theory.

16.1 Oriented Intersection Theory

When two manifolds $X, Z \subset Y$ intersect transversely, choose some $x \in X \cap Z$. Then, observe that we have a short exact sequence

$$0 \to T_x(X \cap Z) \to T_x(X) \oplus T_x(Z) \to T_x(Y) \to 0.$$

By Exercise 15.1, this allows us to induce an orientation on $X \cap Z$, given a specific orientation on X, Z, and Y. In the particular case that X and Z are of complementary dimension, $T_x(X \cap Z) = 0$, so this short exact sequence becomes an isomorphism $T_x(X) \oplus T_x(Z) \cong T_x(Y)$. Locally, we can choose coordinates around x so that the projection onto X is just the first k dimensions, and the projection onto Z is the last ℓ dimensions.

Definition 16.1 (Intersection index)**.** Given $X \pitchfork Z$ in Y of complementary dimension (all oriented), choose some $p \in X \cap Z$. Then, the *intersection index* of p, denoted $\iota_p(X, Z)$, is an integer ± 1 representing the orientation of p in the zero-dimensional vector space $X \cap Z$, induced by the short exact sequence of tangent spaces from transversality. In other words, if $X \oplus Z$ has the same orientation as Y, then $\iota_p(X, Z) = 1$. Otherwise, $\iota_p(X, Z) = -1$.

Note that we can also generalize the above to functions $f : X \to Y$, not just inclusions. The same reasoning holds with some adjustment, as long as $f \pitchfork Z$. We will use this in future definitions.

Definition 16.2 (Oriented intersection number)**.** Suppose that X is compact, and $Z \subset Y$ is a submanifold of complementary dimension, each with an orientation. Given $f : X \to Y$ and $f \pitchfork Z$ in Y, we define the *oriented intersection number* of f with Z in Y to be

$$I(f, Z) = \sum_{p \in f^{-1}(Z)} \iota_p(f, Z).$$

When $f : X \to Y$ is an embedding, we often abbreviate this as $I(X, Z)$.

This is a refinement of the $I_2(f, Z)$ definition in Proposition 11.2, when we have orientations. As before, we're going to check out the most extreme cases first and see what happens. Note that when $\dim X = \dim Y$ and $p \in f^{-1}(Z)$, we have $\iota_p(X, Z) = 1$ if and only if $\mathrm{d}f_p$ is orientation-preserving.

Definition 16.3 (Degree)**.** If $\dim X = \dim Y$ and $f : X \to Y$ is a function between oriented manifolds, then consider some $y \in Y$ which is a regular value of f. (This exists by Proposition 6.2.) We define the *degree* of f to be $I(f, \{y\}) \in \mathbb{Z}$, and denote it $\deg(f)$.

The definition of degree does not depend on the specific choice of $y \in Y$, since just like in the modulo 2 case, we can draw a path between any two regular values y and y', which preserves the oriented intersection number.

Proposition 16.4 (Oriented moving lemma)**.** *For any $f : X \to Y$ and $Z \subset Y$, where X is compact, and all three manifolds are oriented, there exists some map f' homotopic to f such that $f' \pitchfork Z$ as well. Furthermore, $I(f', Z)$ is independent of the choice of f'.*

Proof. The first part of this lemma, existence, is the same as Proposition 11.2. Next, assume we have two functions $f \pitchfork Z$ and $f' \pitchfork Z$, which are homotopic through $F : I \times X \to Y$ such that $F(0, x) = f(x)$ and $F(1, x) = f'(x)$, and $F \pitchfork Z$. By transversality, $F^{-1}(Z)$ is an oriented, compact 1-manifold with boundary. By Lemma 15.5, the total index on the boundary of $F^{-1}(Z)$ is zero. However, $\partial F^{-1}(Z) = f'(Z) - f(Z)$, so $I(f, Z) = I(f', Z)$, as desired. □

The oriented moving lemma lets us define $I(f, Z)$ even when f is not transverse to Z. With this, we have a full set of tools ported over from the modulo 2 case.

16.2 The Fundamental Theorem of Algebra

Let's show one impressive application of oriented intersection theory, which is proving the fundamental theorem of algebra using purely topological methods (no complex analysis!). In the modulo 2 case, we did not have the tools to work with general polynomials, only odd-degree ones. This time, we will have everything we need.

Proposition 16.5 (Fundamental theorem of algebra)**.** *If $f : \mathbb{C} \to \mathbb{C}$ is a complex polynomial of degree $n \geq 1$, then $f(z) = 0$ for some z.*

Proof. Assume for the sake of contradiction that $f \neq 0$ everywhere. By a growth argument, since $|f(z)| = \Theta(|z|^n)$, we can restrict the domain of f to a sufficiently large circle around the origin, indicating that the winding number of f around the origin is n. However, this circle is the boundary of a disk, and f along this disk intersects transversally with the origin. By an analogue to Proposition 12.4, the winding number must be zero, which is a contradiction. □

As a remark, this has interesting connections to complex analysis. If f is a holomorphic function, then f is orientation preserving whenever $f' \neq 0$. This has interesting applications to complex analysis, such as in the proof of Rouché's theorem, which relates the number of roots of a holomorphic function inside a region to the degree of the function on the boundary.

17 March 29th, 2021

Today, we are at a critical junction in the course. So far we've covered all of the basic theory: manifolds, derivatives, immersions, submersions, orientation, and intersection theory. For the remaining nine lectures, we will be applying this theory to prove interesting facts in topology.

17.1 Intersection Theory Between Functions

So far, we've seen transversality and intersection theory between two manifolds, $X \pitchfork Z$, and between a manifold and a function, $f \pitchfork Z$. The former is a special case of the latter when f is an embedding. We can naturally generalize this further to two functions $f \pitchfork g$.

Given two smooth maps $f : X \to Y$ and $g : Z \to Y$, where $k = \dim X$, $\ell = \dim Z$, $n = \dim Y$, and $k + \ell = n$, we say that $f \pitchfork g$ at a point $(x, z) \in X \times_Y Z$ if

$$\operatorname{im} \mathrm{d}f_x + \operatorname{im} \mathrm{d}g_z = T_y(Y).$$

This is equivalent to the following definition, which allows us to reuse Definition 4.8.

Definition 17.1 (Transversality between functions)**.** Given two smooth maps $f : X \to Y$ and $g : Z \to Y$, we say that $f \pitchfork g$ if and only if $(f \times g) \pitchfork \Delta$, where $\Delta \subset Y \times Y$ is the diagonal manifold.

Under this definition, we can also generalize Definition 16.2, oriented intersection number, to the case of two maps. It can be verified that $I(f, g) = (-1)^\ell \cdot I(f \times g, \Delta)$. This means that for any oriented manifolds X, Y of dimensions n and $2n$, we can define $I(X, X)$ unambiguously.

17.2 Euler Characteristic

Here's a particularly important special case of the idea we introduced.

Definition 17.2 (Euler characteristic)**.** If X is a manifold, let $\Delta \subset X \times X$ be the diagonal manifold in $X \times X$. Then, $I(\Delta, \Delta) \in \mathbb{Z}$ is called the *Euler characteristic* of X and denoted $\chi(X)$.

This is the same Euler characteristic often seen in graph theory and simplicial complexes. It turns out that even the $V - E + F = 2$ formula for polyhedra comes from this!

Exercise 17.1. Show that if $Y' \to Y$ is a d-sheeted covering space, then $\chi(Y') = d \cdot \chi(Y)$.

Here's another interesting fact about intersections, which is not immediately obvious.

Lemma 17.3. *If $n = \dim X$ is odd and Y is a manifold of dimension $2n$, then $I(X, X) = 0$.*

Proof. Recall that $I(X, Z) = (-1)^{k\ell} I(Z, X)$, for any X and Z, since we reverse the order of the bases. Therefore, $I(X, X) = (-1)^{n^2} I(X, X)$. When n is odd, this implies that $I(X, X) = 0$. □

Corollary 17.3.1. *If X is a compact manifold of odd dimension, then $\chi(X) = 0$.*

Example 17.4. If S^1 is a circle, then the diagonal manifold Δ in $T = S^1 \times S^1$ is a loop on the torus. By rotating the torus slightly, we can deform Δ to a new loop that does not intersect the old one. Therefore, $I(\Delta, \Delta) = 0$ in T, which is consistent with Corollary 17.3.1.

Example 17.5. The Euler characteristic of S^2 is equal to 2. Any rotation f of S^2 will have exactly two fixed points, and these fixed points in $\Gamma_f = \{(x, f(x)) \mid x \in S^2\}$ intersect transversally with the diagonal $\Delta \subset S^2 \times S^2$.

Example 17.6. Consider the Euler characteristic of the torus $T = S^1 \times S^1$. Then, $\chi(T)$ is equal to the intersection number $I(\Delta, \Delta)$ on $T \times T$. There exists a simple deformation of the identity map $\mathrm{id} : T \to T$ to a map simply by a translating $x \mapsto x + v$ for some $v \in \mathbb{R}^2$, then factoring through $T \cong \mathbb{R}^2/\mathbb{Z}^2$. In this case, the image Γ_v is disjoint from T, so $\chi(T) = 0$.

Alternatively, instead of doing the above calculation for the torus, we will see later on that for any compact manifolds X and Y, $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$. Let's see something cool that generalizes the above "translation" strategy in S^1 for computing the Euler characteristic.

Definition 17.7 (Lie group)**.** A *Lie group* G is a differentiable manifold with smooth maps $m : G \times G \to G$ and $i : G \to G$ that satisfy the axioms of a group (multiplication, inversion).

A Lie group can be thought of as a set of points with two separate but compatible algebraic structures, being both a manifold and a group. Examples include $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{O}(n), \mathrm{U}(n)$, and others.

Lemma 17.8. *A compact Lie group G has Euler characteristic $\chi(G) = 0$.*

Proof. Apply a group action $t_g(h) = m(g, h)$ for some $g \in G$ that is not the identity element. This has no fixed points by the group axioms, meaning the graph Γ_{t_g} is disjoint from Δ (and therefore intersects transversally with it), so $I(\Delta, \Delta) = 0$ for Δ being the diagonal of $G \times G$. □

As an immediate corollary, S^2 cannot be given the structure of a Lie group. Eventually, ideas in this vein will help us arrive at the Lefschetz fixed point formula, which we'll see next lecture.

18 April 5th, 2021

Today we cover Lefschetz fixed-point theory.

18.1 Lefschetz Fixed Points

Recall that we previously defined oriented intersection numbers between two submanifolds, and we used this to introduce the Euler characteristic $\chi(X) = I(\Delta, \Delta)$, where $\Delta = \{(x, x) \mid x \in X\}$ is the diagonal in $X \times X$. To generalize the Euler characteristic, consider a smooth map $f : X \to X$, and the graph of f is a set $\Gamma_f = \{(x, f(x)) \mid x \in X)\}$

Definition 18.1 (Lefschetz number)**.** For a smooth map $f : X \to X$, the *Lefschetz number* of f is the oriented intersection number between Γ_f and Δ in $X \times X$. In other words, $L(f) = I(\Gamma_f, \Delta)$.

Additionally, we call f a *Lefschetz map* if $\Gamma_f \pitchfork \Delta$. When this is the case, the Lefschetz number $L(f)$ is equal to the number of fixed points of f, but counting sign of each fixed point. Keeping this in mind, we can then ask what conditions are necessary for $\Gamma_f \pitchfork \Delta$, and how to compute $\iota_{(p,p)}(\Gamma_f, \Delta)$ at each fixed point $p \in X$. This requires some linear algebra.

Recall that both $T_{(p,p)}(\Delta)$ and $T_{(p,p)}(\Gamma_f)$ are vector subspaces of dimension n in $T_{(p,p)}(X \times X)$, where $X \times X$ is a manifold of dimension $2n$. For these two subspaces to span the tangent space, their intersection must have dimension zero; in other words, the subspaces are independent. Therefore, $\Gamma_f \pitchfork \Delta$ precisely when $\mathrm{d}f_x$ has no eigenvalue equal to 1 at each fixed point $x = (p, f(p))$. Furthermore, the sign of a Lefschetz fixed point is equal to the sign of $\det(\mathrm{d}f_x - I)$, where we define an inner product on the tangent space through inclusion in $\mathbb{R}^N$.

Example 18.2 (Lefschetz fixed points, $n = 1$)**.** In one dimension, this is especially easy to visualize. The graph Γ_f is simply a smooth curve in $\mathbb{R}^2$ that passes the vertical line test. When $f'(x) > 1$ at a fixed point, f has higher slope than the diagonal line $y = x$, and the sign of the fixed point is positive. When $f'(x) < 1$, the sign is negative, and when $f'(x) = 1$, we do not have transversality, so x is not a Lefschetz fixed point.

Example 18.3 (Lefschetz fixed points, $n = 2$)**.** In dimension 2, we have five cases. When both eigenvalues $\alpha_1, \alpha_2 > 1$, we have a repelling fixed point. When $\alpha_1, \alpha_2 < 1$, the fixed point is attracting.[14] Meanwhile, if $\alpha_1 > 1$ and $\alpha_2 < 1$, we have a saddle point. The other two cases are when the eigenvalues are complex, representation a rotation, and when one of the eigenvalues is 1.

Let's see the previous example of Lefschetz fixed points for $n = 2$ in practice. Suppose that we have a double torus, i.e., a torus with two holes. We vertically orient it so that one hole is directly above the other one, according to some "height function" similar to in Example 4.4. If we were to pour some heavy liquid ("hot fudge") on the manifold, then at each point there is a unique direction of steepest descent where the liquid will fall, except at the six critical points. Let our function f be the position of the fudge after a small amount of time Δt. The top-most point is repelling, as the liquid moves down away from it. Similarly, the bottom-most point is attracting. All four points in the middle are saddle points.

To compute the Euler characteristic of the torus, observe that f is homotopic to the identity map by simply taking all intermediate time intervals in $[0, \Delta t]$. Therefore, $I(\Delta, \Delta) = I(\Gamma_f, \Delta)$, which equals the sum of the local Lefschetz numbers at each fixed point. The attracting and repelling points contribute $+1$ each, while the saddle points contribute -1 each, so $\chi = 2 - 4 = -2$.

[14] It's not necessarily attracting in the dynamical system sense; what if the eigenvalue is less than 0? But attraction is still a good analogy, and for our purposes, the visual examples will all have positive eigenvalues.

18.2 Ubiquity of Lefschetz Maps

In order to make the Lefschetz number $L(f) = I(\Gamma_f, \Delta)$ well-defined in all cases, we need to make sure that the intersection numbers are valid.

Proposition 18.4. *If $f : X \to X$ is a smooth map that is not Lefschetz, then there exists some small deformation f' that is homotopic to f such that f' is Lefschetz.*

Proof. A proof of this statement is given in the book. It is analogous to the moving lemma. □

Now, motivated by the example of easily computing the Euler characteristic for a complex manifold like the double torus, we state a version of the Lefschetz fixed-point theorem.

Definition 18.5 (Fixed-point index)**.** Suppose that x is an isolated fixed point of $f : X \to X$, where $\dim X = n$. For a small ball U around x, we have a coordinate chart $\phi : U \to \mathbb{R}^n$, and we define a function $g : S^{n-1} \to S^{n-1}$ by

$$g(\phi(x)) = \frac{\phi(x - f(x))}{\|\phi(x - f(x))\|}.$$

Then, the *index* of x as a fixed point, denoted $L_x(f)$, is defined as $\deg(g)$.

Proposition 18.6 (Lefschetz fixed-point formula)**.** *If the set of fixed points of f is isolated, then*

$$L(f) = I(\Gamma_f, \Delta) = \sum_{x : f(x) = x} L_x(f).$$

This is convenient because it allows us to compute intersection numbers in some cases without having to construct explicit deformations that make the map transversal. Generally, we have three strategies for computing the Lefschetz number of a map, in increasing order of generality:

1. If all fixed points of f are Lefschetz, then we can compute L_f by simply summing over the intersection index at each fixed point.

2. If the fixed points of f are isolated (and therefore finite in number), we can compute $L(f)$ using Proposition 18.6.

3. Otherwise, $L_f = I(\Gamma_f, \Delta)$ can be computed by taking a deformation.

In the next lecture, we will discuss vector fields and the Hopf index.

19 April 7th, 2021

Today we discuss vector fields on smooth manifolds.

19.1 Vector Fields

Recall Definition 7.9, where if $X \subset \mathbb{R}^N$ is a manifold of dimension n, then the tangent bundle TX is a manifold of dimension $2n$ corresponding to the set

$$TX = \{(x, v) \in X \times \mathbb{R}^N : v \in T_x X\}.$$

The projection map from the tangent bundle TX to its first component in X is a submersion, with fibers equal to the tangent spaces at each point $x \in X$. Locally, because X looks like $\mathbb{R}^n$ via a coordinate chart at each point, the corresponding section of the tangent bundle is diffeomorphic to a product vector space $\mathbb{R}^n \times T_x X$.

Definition 19.1 (Vector field)**.** A *vector field* v is a smooth map $v : X \to TX$ such that $\pi \circ v = \mathrm{id}$, where $\pi : TX \to X$ is the projection map $(x, v) \mapsto x$ onto the base point in the tangent bundle.

Intuitively, a vector field is simply a vector-valued function on the manifold that lies in the tangent plane at every point, while also varying smoothly along the manifold. It is a **prescription for motion**. A basic fact about vector fields is that given a vector field v on $\mathbb{R}^n$, there exists a unique function $F : I \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\frac{\partial F}{\partial t}(t, x) = v(F(t, x))$.[15] Such a function would be consistent in the intersection of any two coordinate charts on a manifold X, so for any manifold X and vector field v, we can describe the global motion of points by a smooth flow $F : I \times X \to X$.

A key fact about vector fields is that the three statements of Lefschetz fixed-point theory from last lecture have analogues to vector fields. Given a vector field v, we say that v has *non-degenerate zeros* if the derivative of v at each zero is nonsingular. Then, the Lefschetz number $L(v)$ is simply the sum of ± 1 indices at each zero. More generally, if v has *isolated zeros*, then there is an induced map $h : S^{n-1} \to S^{n-1}$ defined by $h(x) = \frac{v(x)}{\|v(x)\|}$. We can then write that $L_x(v) = \mathrm{ind}_v(x) = \deg(h)$, which is consistent with the index of the flow $F_t : X \to X$ corresponding to v.

Proposition 19.2 (Poincaré-Hopf theorem)**.** *If v is a vector field on X with isolated zeros, then*

$$\sum_{x : v(x) = 0} \mathrm{ind}_x(v) = \chi(X).$$

The contrapositive is that if $\chi(X) \neq 0$, then every vector field on X must have a zero.

Proof. Consider a flow $F(t, x)$ corresponding to movement along the vector field v. The flow gives us a natural homotopy between $x = F(0, x)$ and $f(x) = F(1, x)$, so we have that

$$\sum_{x : v(x) = 0} \mathrm{ind}_x(v) = L(f) = I(\Gamma_f, \Delta) = I(\Delta, \Delta) = \chi(X).$$

□

The contrapositive of the above immediately implies a famous theorem in topology.

Corollary 19.2.1 (Hairy ball theorem)**.** *There is no smooth, nonvanishing vector field on S^2.*

Proof. Observe that $\chi(S^2) = 2 \neq 0$. □

[15] This is the existence and uniqueness of solutions to first-order ODEs.

19.2 Motivating Differential Forms

Now, we briefly introduce the notion of differential forms as covered in Chapter 4 of [GP10]. The broad goal of this chapter is to define a notion of integration on manifolds, similar to how we have Riemann and Lebesgue integration on the real line.

Let's first start with the most basic example, where we want to integrate a function f over a cell in $\mathbb{R}^n$. This is simple; we just use the Riemann integral once again (like over $\mathbb{R}$) but iterate over the dimensions. Things get significantly more complex once we start adding curvature. How might we integrate a function $f : X \to \mathbb{R}$ over a 1-manifold X? One attempt would be to find a diffeomorphism $\phi : [a, b] \to X \subset \mathbb{R}^N$ so that

$$\int_X f = \int_a^b \phi^* f.$$

However, this **depends on the parameterization** ϕ. For example, when scaling ϕ from $[0, 1] \to X$ to $[0, 2] \to X$, we would naively double the value of the integral. To fix this apparent paradox, we need to introduce the idea of a differential form (the formal meaning of the "dx" symbol from calculus) and pullbacks, which will be discussed next lecture.

20 April 12th, 2021

In the last unit of the course, we will be covering integration on manifolds. The plan is to first discuss how to perform integration on 1-manifolds, then introduce higher exterior powers on a vector space. We'll use the exterior algebra to perform integration on n-manifolds.

20.1 Integration on $\mathbb{R}^n$

The standard Riemann integral of a function f on some interval $[a, b] \subset \mathbb{R}$ is defined as the area under the curve of the function, or a Riemann sum

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\forall i: x_{i+1} - x_i \to 0} \sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i),$$

where $a = x_0 < t_0 < x_1 < t_1 < \cdots < t_{n-1} < x_n = b$ is a *tagged partition* of the interval. The basic theorem is that if f is continuous almost everywhere, then this Riemann sum is well-defined as a limit. Furthermore, we can generalize the Riemann integral to the *Lebesgue integral*, which allows us to integrate any *measurable* function.[16]

To generalize the Riemann integral to bounded subsets of $\mathbb{R}^n$, we simply divide up the region into a collection of small cells and use the Riemann sum once again, this time in multiple dimensions. We can also generalize this naturally using Lebesgue integration by taking the product measure. Although this has been simple so far in Euclidean space, it gets more complicated as soon as we start bringing in manifolds.

20.2 Integration Over a 1-Manifold

Suppose that X is a compact, connected, n-manifold with boundary, and let's start by considering the case when $n = 1$. If $f : X \to \mathbb{R}$ is a continuous function, we want to define $\int_X f$ analogously. One way to to naively attempt this would be to introduce a parameterization $\phi : [a, b] \to X$ and define

$$\int_X f = \int_a^b f \circ \phi.$$

However, this depends heavily on the choice of parameterization ϕ. For example, if the parameterization is twice as "slow," then the value of the integral will be twice as large. The solution to this apparent paradox is to smoothly associate each element $x \in X$ with an element of the cotangent space $T_x^*(X)$. We call this a 1-*form.*

Definition 20.1 (1-form)**.** A 1-*form* is the dual of a vector field (Definition 19.1), defined as a smooth[17] map $\omega : X \to T^*X$ that associates an element of the cotangent space $\omega(x) \in T_x^*X$ to each point $x \in X$.

Recall that tangent vectors and vector fields **push forward**, in the sense that given a smooth map $f : X \to Y$ and a vector field $v : X \to TX$, we can induce a canonical pushforward vector field $\mathrm{d}f(x) : Y \to TY$. The basic fact about 1-forms is that they **pull back**. Given a 1-form $\omega : Y \to T^*Y$, we have a pullback form $f^*\omega : X \to T^*X$ defined by

$$f_x^*\omega(v) = \omega(y)(\mathrm{d}f_x(v)).$$

[16]For example, the indicator function on $\mathbb{Q}$ is not Riemann integrable, but it is Lebesgue integrable.

[17]We will assume that differential forms are smooth in this class, but integration works with laxer assumptions.

Now, suppose that Y is a manifold and ω is a 1-form on Y. Then, if $X \subset Y$ is a connected 1-dimensional submanifold with boundary, we can parameterize X by $\phi : [a, b] \to X$. We then define the integral of a 1-form to be

$$\int_X \omega = \int_{[a,b]} \phi^* \omega = \int_a^b f(t)\, \mathrm{d}t,$$

where $\phi^*\omega$ is a 1-form on $\mathbb{R}$ written in the notation $f(t)\,\mathrm{d}t$, and $\mathrm{d}t \in T_x(\mathbb{R})$ is the element of the cotangent space that takes the positive unit vector to 1. The key fact is that this definition of integration by pulling back a 1-form onto an interval (1-cell) is **invariant** under change of parameterization, which is a consequence of the chain rule.

Note. A more concrete way of defining smoothness would be to take a basis $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ of $T_x^*\mathbb{R}^n$ for all $x \in X$, which simply are the dual vectors of the standard basis for $\mathbb{R}^n$. Any 1-form ω on $X \subset \mathbb{R}^n$ can be written in this basis as

$$\omega(x) = \sum_{i=1}^{n} \psi_i(x)\, \mathrm{d}x_i,$$

and ω is smooth if and only if the components $\psi_i : X \to \mathbb{R}$ are all smooth functions.

We now introduce the exterior derivative in the simplest setting. Consider a smooth function $f : X \to \mathbb{R}$, which is a 0-form, and this has a derivative $\mathrm{d}f_x : T_xX \to \mathbb{R}$. By the definition of the cotangent space, we have $\mathrm{d}f_x \in T_x^*X$. Therefore, $\mathrm{d}f$ can actually be viewed as a 1-form on X. If Y is a manifold and $X \subset Y$ is a 1-manifold with boundary, then

$$\int_X \mathrm{d}f = \int_{\partial X} f.$$

In concrete terms, if we write this over the reals, we have $\int_a^b \mathrm{d}f = f(b) - f(a)$, which is precisely the fundamental theorem of calculus. We will see by the end of this course how to generalize this idea via *Stokes' theorem* to integrals of differential forms over manifolds of arbitrary dimension.

21 April 14th, 2021

Today we generalize our definition of 1-forms to higher k-forms using the wedge product, and we introduce the exterior calculus on differential forms.

21.1 Differential Forms

First, recall the definition of the exterior algebra $\bigwedge^k V^*$ on a finite-dimensional vector space. Since we assume familiarity with the tensor product and exterior algebra in this course, we will simply write down the most general definition of $\bigwedge^k$ as a universal property.

Definition 21.1 (Exterior algebra)**.** The *exterior algebra* $\bigwedge^k V^*$ of order k on the dual of a vector space V is a vector space consisting of all skew-symmetric k-linear forms on V.

If $n = \dim V$, then the dimension of $\bigwedge^k V^*$ is $\binom{n}{k}$. This is because if $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ is a basis for the original vector space V^*, then we can explicitly write down a basis for $\bigwedge^k V^*$ by taking

$$\{\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k} : 1 \le i_1 < \cdots < i_k \le n\}.$$

Furthermore, the exterior power $\bigwedge^k$ is a *functor*, in the sense that any map $f : V^* \to V^*$ can be lifted to a map $f : \bigwedge^k V^* \to \bigwedge^k V^*$ by simply applying f to each of the atomic terms within the wedge products. The last fact that we will use is that the wedge product is a map

$$\wedge : \bigwedge^k V^* \times \bigwedge^\ell V^* \to \bigwedge^{k+\ell} V^*,$$

such that for any k-tensor α and ℓ-tensor β, we have $\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha$. We are now finally ready to define differential forms.

Definition 21.2 (k-form)**.** A differential *k-form* on a manifold X is a smooth map ω taking each point $x \in X$ to an element of the exterior algebra $\omega(x) \in \bigwedge^k T_x^* X$ on the cotangent space at x.

Just as we saw with 1-forms, the key fact about k-forms is that they **pull back**. Given a smooth map $f : Y \to X$ and a k-form ω on X, we have a derivative $\mathrm{d}f_x : T_y Y \to T_{f(y)} X$. This naturally induces a pullback map $f^* : \bigwedge^k T_{f(y)}^* X \to \bigwedge^k T_y^* Y$ between the corresponding cotangent spaces by

$$f^*(\omega)(v_1, \ldots, v_n) = \omega\big(\mathrm{d}f_y(v_1), \ldots, \mathrm{d}f_y(v_n)\big).$$

Note that if $X \subset \mathbb{R}^n$ is an open subset, then we have a standard basis for T_x^* given by $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$. This allows us to write a general k-form as a summation $\sum_I f_I(x)\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$. A differential k-form is formally defined to be smooth if and only if each of its components $f_I : X \to \mathbb{R}$ is a smooth function. In the special case when $k = n$, we can write

$$\omega(x) = f(x)\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n,$$

so the n-th exterior power of a vector space is one-dimensional. (In particular, this is one way of defining the *determinant* naturally.)

21.2 Integrating Differential Forms

To define integration on differential forms, we pull them back to an open subset of Euclidean space.

Definition 21.3 (Integration of k-forms)**.** Suppose that $X \subset \mathbb{R}^n$ is a bounded open subset and ω is an n-form on X. Then, if $\omega(x) = f(x)\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, then we define

$$\int_X \omega = \int_X f(x)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

where the right-hand side is a standard Riemann or Lebesgue integral on Euclidean space. Furthermore, if $X \subset \mathbb{R}^N$ is an n-manifold with open neighborhood $U \subset X$, and we have a diffeomorphism $f : \Omega \to U$ where $\Omega \subset \mathbb{R}^n$, then

$$\int_U \omega = \int_\Omega f^*\omega.$$

To generalize to the integration of ω on the entire n-manifold X, we apply a *partition of unity*. Let $\{f_\alpha\}$ be a collection of smooth functions $f_\alpha : X \to \mathbb{R}$ such that the support of f_α is on an open neighborhood $U_\alpha \cong \mathbb{R}^n$, and $\sum_\alpha f_\alpha = 1$. Then, we say that

$$\int_X \omega = \sum_\alpha \int_{U_\alpha} f_\alpha \omega.$$

In practice, people don't often use the theoretical definition above because explicitly computing partitions of unity is not very ergonomic. Instead, we often "cheat" by breaking up the manifold X into a collection of nice disjoint coordinate patches. For example, a sphere could be cut along the equator to produce two circles, or it could be cut along a meridian to produce a rectangle in spherical coordinates.

21.3 The Exterior Derivative

Let's introduce one more construction that has a big role to play in the analysis of differential forms and integration on manifolds. First, we use $A^k(X)$ as a notation for the vector space of differential k-forms on a manifold X.[18] Then, if $f : X \to \mathbb{R}$ is a smooth map (i.e., it is a 0-form $f \in A^0(X)$), then we define $\mathrm{d}f : TX \to \mathbb{R}$ to be the 1-form on X given by

$$\mathrm{d}f(x, v) = \mathrm{d}f_x(v) \in T_{f(x)}\mathbb{R} = \mathbb{R}.$$

This induces a linear map $\mathrm{d} : A^0(X) \to A^1(X)$. The analogy to this in higher dimensions is a map $A^k(X) \to A^{k+1}(X)$ for any degree $k \geq 0$.

Definition 21.4 (Exterior derivative)**.** On a manifold X, the *exterior derivative* is a linear map $\mathrm{d} : A^k(X) \to A^{k+1}(X)$. If $\omega \in A^k(X)$ is a k-form, and we write it in local coordinates $x_1, \ldots, x_n$ by

$$\omega = \sum_I f_I \,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k},$$

then we define the exterior derivative $\mathrm{d}\omega \in A^{k+1}(X)$ in local coordinates by

$$\mathrm{d}\omega = \sum_I \mathrm{d}f_I \,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}.$$

[18] Gaurav notes that this is also commonly notated as $\Omega^k(X)$.

The key properties of the exterior derivatives are as follows.

- **Commutes with pullback:** If $f : X \to Y$ is smooth and $\omega \in A^k(Y)$, then $\mathrm{d}f^*\omega = f^*\,\mathrm{d}\omega$.
- **Square is zero:** $\mathrm{d}^2 = 0$, or more specifically, if $\omega \in A^k(X)$, then $\mathrm{d}(\mathrm{d}\omega) \in A^{k+2}(X) = 0$.
- **Product rule:** If $\alpha \in A^k(X)$ and $\beta \in A^\ell(X)$, then $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta$.

These properties are enough to completely characterize the exterior derivative.

Note. In typical treatments of vector calculus on manifolds in $\mathbb{R}^3$, such as Math 21 and introductory physics classes, working within the standard basis of Euclidean space (with inner product) induces a canonical isomorphism between $\mathbb{R}$ and its dual $\mathbb{R}^*$. Therefore, many courses will choose to explicitly write down k-forms as vector fields, rather than introduce new language. All of the standard vector calculus operations ∇A, $\nabla \cdot V$, and $\nabla \times V$ are physical manifestations of the exterior derivative to the special case of k-forms in $\mathbb{R}^3$.

22 April 19th, 2021

We start by reviewing the topics that we covered. So far in this section on differential forms, we've seen a lot of algebra (in the exterior product) and analysis (integration), but this is a topology class! Today, we'll see the relationship between differential forms and topology.

22.1 Recap of Differential Forms

To review what we previously covered, a 1-form is a smooth assignment of a linear form $f \in T_x^*(X)$ to each point $x \in X$ of a manifold. This is essentially the dual of a tangent vector field. A k-form is the generalization of this, which is the assignment of a skew-symmetric k-multilinear map to each point $x \in X$ of a manifold. If $\{x_1, \ldots, x_n\}$ is a local basis for X, then we can write differential forms in terms of a local basis $\mathrm{d}x_I$ for each $I \subset \{1, \ldots, n\}$.

Differential forms pull back. If $f : X \to Y$ is a smooth map between manifolds taking some point $p \in X$ to $q = f(p) \in Y$, then there exists a derivative $\mathrm{d}f_p : T_pX \to T_qY$, which induces a dual map $\mathrm{d}f_p^* : T_qY \to T_pX$. This naturally lifts with the exterior product to produce a linear map

$$\bigwedge^k \mathrm{d}f_p^* : \bigwedge^k T_qY \to \bigwedge^k T_pX.$$

Then, if $\omega \in A^k(Y)$, i.e., it is a k-form on Y, then we define $f^*\omega \in A^k(X)$ to be the pullback k-form

$$(f^*\omega)(p) = \bigwedge^k \mathrm{d}f_p^*(\omega(q)).$$

Differential forms can be integrated. To integrate a differential k-form ω over a k-manifold, we can pull back the manifold (through partitions of unity) to a collection of k-cells in Euclidean space $\mathbb{R}^k$, in which $f^*\omega$ becomes a volume form. Then, integration can proceed by the standard Riemann or Lebesgue definitions in flat space.

Exterior derivatives. The exterior derivative is a linear map $\mathrm{d} : A^k(X) \to A^{k+1}(X)$. It obeys the product rule with some sign corrections, its square is $\mathrm{d}^2 = 0$, and one can verify that it commutes with the pullback like so:

$$\begin{array}{ccc} A^k(Y) & \xrightarrow{f^*} & A^k(X) \\ \mathrm{d}\downarrow & & \downarrow\mathrm{d} \\ A^{k+1}(Y) & \xrightarrow{f^*} & A^{k+1}(X). \end{array}$$

Note that the exterior derivative generalizes the notions of curl, divergence, and gradient. By applying this construction in dimensions 2 and 3, we arrive at familiar theorems in vector calculus, such as $\nabla \cdot (\nabla \times V) = 0$ and $\nabla \times (\nabla A) = 0$ from $\mathrm{d}^2 = 0$.

22.2 De Rham Cohomology

Notice that the property $\mathrm{d}^2 = 0$ of the exterior derivative implies that if X is any n-manifold, then we have a sequence of vector spaces and maps

$$0 \to A^0(X) \xrightarrow{\mathrm{d}_0} A^1(x) \xrightarrow{\mathrm{d}_1} A^2(X) \xrightarrow{\mathrm{d}_2} \cdots \xrightarrow{\mathrm{d}_{n-2}} A^{n-1}(X) \xrightarrow{\mathrm{d}_{n-1}} A_n(X) \to 0,$$

such that for each i, we have $\mathrm{im}(\mathrm{d}_i) \subset \ker(\mathrm{d}_{i+1})$. This construction is called a *cochain complex*, and it induces a *cohomology* as such.

Definition 22.1 (De Rham cohomology)**.** For a smooth manifold X of dimension n, the *de Rham cohomology* consists of a collection of cohomology groups $H^k_{\mathrm{dR}}(X)$ for each $0 \le k \le n$. We call a differential form $\omega \in A^k(X)$ *closed* if $\mathrm{d}\omega = 0$ and *exact* if $\omega = \mathrm{d}\alpha$ for some $\alpha \in A^{k-1}(X)$. The k-th cohomology group is the quotient vector space of closed forms modulo exact forms in $A^k(X)$.

Note. We use the notation $H^k_{\mathrm{dR}}(X)$ for the de Rham cohomology, but it is the only cohomology group that we will describe in this course. In a slightly broader point of view, algebraic topology has many different cohomology constructions. If you write H_k, most algebraic topologists will assume that you are talking about *simplicial homology* or *singular homology*, which are different but related constructions.

The association $X \rightsquigarrow H^k_{\mathrm{dR}}(X)$ is a contravariant functor between manifolds and vector spaces. Or, in less abstract language, cohomology groups pull back.

Lemma 22.2 (Poincaré lemma)**.** *If $X \subset \mathbb{R}^N$ is a* star-shaped *n-manifold, meaning that all line segments coming from 0 to a point $x \in X$ are contained within X, then the cohomology of X is trivial.*

Corollary 22.2.1. *Every conservative (curl-free) vector field in $\mathbb{R}^3$ is the gradient of some potential. Likewise, every divergence-free vector field is the curl of some vector potential.*

In particular, we can speak slightly more generally through topology. If $f, g : X \to Y$ are homotopic, then the contravariant functors f^* and g^* mapping the de Rham cohomology groups $H^k_{\mathrm{dR}}(Y) \to H^k_{\mathrm{dR}}(X)$ are equal.[19] In particular, if X is contractible, then $H^k_{\mathrm{dR}}(X) = 0$ for all $k > 0$. Here's how we might use this fact in a topological setting.

Example 22.3. Suppose that we wanted to show that $\mathbb{R}^2$ is not homotopy equivalent to $\mathbb{R}^2 \setminus \{0\}$. One way of doing this is to exhibit a closed 1-form on $\mathbb{R}^2 \setminus \{0\}$ that is not exact. For example, one can take the dual of the vector field

$$(x, y) \longmapsto \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

This implies that $H^1_{\mathrm{dR}}(\mathbb{R}^2 \setminus \{0\}) \neq 0$, so $\mathbb{R}^2 \setminus \{0\}$ is not contractible, but $\mathbb{R}^2$ is contractible.

Another important fact that we'll need to talk about cohomology theory is Stokes' theorem.

Proposition 22.4 (Stokes' theorem)**.** *If X is an n-manifold with boundary, and $\omega \in A^{n-1}(X)$ is an $(n-1)$-form with compact support, then*

$$\int_X \mathrm{d}\omega = \int_{\partial X} \omega.$$

Now Joe Harris provides a thought experiment to motivate (co)homology, for students who are unfamiliar with simplicial homology from other topology classes. Consider the circle $S^1 \subset \mathbb{R}^2$. This is the boundary of the closed unit disk in $\mathbb{R}^2$, but it is not the boundary of any submanifold of $\mathbb{R}^2 \setminus \{0\}$. Then the challenge is to find, for a given manifold X, the structure

$$\frac{\{\text{submanifolds } Y \subset X \text{ of dimension } k\}}{\{\text{boundaries of submanifolds } Z \subset X \text{ of dimension } k+1\}}.$$

This is a topological invariant, and we can give it an algebraic group structure (modulo a few caveats) using cohomology.

[19] Gaurav phrases this as, the de Rham cohomology functor factors through the smooth homotopy category.

23 April 21st, 2021

Today we will wrap up the content of [GP10] that we will be covering in this course. This is the last lecture from the textbook. The goal of today's lecture is to discuss volumes and degrees, the Gauss-Bonnet theorem[20] (from differential geometry), and cohomology. Next week, the plan is to introduce the modern treatment of manifolds and provide a bridge from the classical 19$^{\text{th}}$ century definition. We will also cover Riemannian manifolds, i.e., smooth manifolds endowed with a metric on the tangent space.

23.1 Degrees and Volumes

Recall that if $f : X \to Y$ is a map of compact, oriented k-manifolds, then the *degree* of f is the oriented intersection number $\deg(f) = I(f, \{y\})$ for any $y \in Y$. Then, the crucial invariant about integration of pullbacks is as follows.

Proposition 23.1. *If $f : X \to Y$ is a smooth map and ω is a k-form on Y, then*

$$\deg(f) \cdot \int_Y \omega = \int_X f^* \omega.$$

Proof. The basic idea is to consider the preimage of some open neighborhood U in Y around some regular value $y \in Y$. This consists of a set of disjoint open subsets of X, each diffeomorphic to $U \subset Y$. Each of these open subsets are covered with a positive or negative index, depending on whether the map is locally orientation-preserving or not. Then, we can cover the complement of the critical locus by open subsets, using a partition of unity to finish our proof. □

In practice, this proposition means that we need to be careful to keep track of orientation sign when integrating a differential form via pullback to a more convenient subset of $\mathbb{R}^k$.

Definition 23.2 (Volume form)**.** Given an oriented manifold $X \subset \mathbb{R}^N$ of dimension k, the *volume form* ν_X on X assigns a k-linear map to each element $x \in X$ such that

$$\nu_x(v_1, \ldots, v_k) = 1$$

if $v_1, \ldots, v_k$ form a positively-oriented orthonormal basis of $T_x X$.

As a consequence of the above definition, if $w_1, \ldots, w_k \in T_x X$ is any other ordered basis, then

$$\nu_X(w_1, \ldots, w_k) = \det(A) \cdot \nu_X(v_1, \ldots, v_k),$$

where A is the change-of-basis matrix taking $v_i \mapsto w_i$ for each $1 \leq i \leq k$. Therefore, whenever we have a manifold embedded in Euclidean space (later, we will see that the manifold simply requires a Riemannian metric), there is a natural volume form on that manifold. We say that the *volume* of a compact oriented submanifold $X \subset \mathbb{R}^N$ is the integral of its volume form,

$$\text{vol}(X) = \int_X \nu_X.$$

Note that the volume form is not preserved under diffeomorphism. Because of this, we can measure the extent to which the volume expands or contracts.

[20] The Gauss-Bonnet theorem is not a purely topological fact, since it depends on a metric connection, but working with submanifolds in Euclidean space gives us a Riemannian metric for free.

Definition 23.3 (Jacobian)**.** If $f : X \to Y$ is a smooth map between compact oriented submanifolds of Euclidean space, then the *Jacobian*[21] of f is defined to be the unique function $J(f) : X \to \mathbb{R}$ such that

$$f^*\nu_Y = J(f)\nu_X.$$

In local coordinates, this is the determinant of the "Jacobian matrix" from vector calculus.

23.2 Curvature and the Gauss-Bonnet Theorem

Suppose that $X \subset \mathbb{R}^{k+1}$ is a k-dimensional hypersurface. In other words, X is a compact oriented manifold. There is a natural map $g : X \to S^k$ providing the direction of outward unit normal vector to X at each point $x \in X$; this is sometimes called the *Gauss map*. Then, we have the following definition of intrinsic Gaussian curvature.

Definition 23.4 (Curvature)**.** The *curvature* κ of X at a point $x \in X$ is given by the Jacobian of the Gauss map, $J(g)(x)$.

For example, in $\mathbb{R}^3$, spheres have constant positive curvature, while hyperboloids have constant negative curvature. A flat plane would have zero curvature, since g does not change. A classic, beautiful theorem from differential geometry relates curvature to the Euler characteristic.

Proposition 23.5 (Gauss-Bonnet theorem)**.** *If X is a compact, orientable k-manifold without boundary for k even, then*

$$\int_X \kappa \cdot \nu_X = \frac{1}{2}\gamma_k \cdot \chi(X),$$

where $\gamma_k = \mathrm{vol}(S^k)$, and $\chi(X)$ is the Euler characteristic of X.

Proof. Note that $\int_X \kappa$ is sometimes written as a shorthand for $\int_X \kappa \cdot \nu_X$, as an abuse of notation when the volume form is clear from context. With that out of the way, we can calculate

$$\begin{aligned}\int_X \kappa &= \int_X \kappa\nu_X \\ &= \int_X J(g) \cdot \nu_X \\ &= \int_X g^*\nu_{S^k} \\ &= \deg(g) \cdot \int_{S^k} \nu_{S^k}.\end{aligned}$$

The latter factor is simply the volume of the k-sphere S^k. Therefore, it remains to show that the degree of the Gauss map $\deg(g)$ is half of the Euler characteristic of X. We apply Sard's theorem to $g : X \to S^k$, which allows us to choose some $a \in S^k$ such that both a and $-a$ are regular values of g. Then, define a vector field v on X so that $v(x)$ is the projection of a onto T_xX.

The zeros of v are precisely the locus of $x \in X$ such that $T_xX \perp a$, or equivalently, $g(x) = \pm a$. Since $\pm a$ are both regular values of g, the total number of such points is $2\deg(g)$. Then, by the Poincaré-Hopf theorem (Proposition 19.2), we conclude that $2\deg(g) = \chi(X)$. ☐

The reason why the Gauss-Bonnet theorem is so surprising and interesting is that it relates a global property, the Euler characteristic (invariant under diffeomorphism), to a notion of curvature that depends on the local geometry of the manifold.

[21] This is also sometimes called the *Jacobian determinant*.

Note. What goes wrong with this proof when k is odd? The problem is that the sum of intersection indices ends up having opposite points cancel each other out, as the degree of the antipodal map is 0 for even k. This is because the antipodal map reverses orientation.

Note. Classically, the Gauss-Bonnet theorem is often stated specifically for the $k = 2$ case and has an extra term for manifolds with boundary. In this case, the statement of the theorem is

$$\int_X \kappa \cdot \nu_X + \int_{\partial X} k_g \cdot \nu_{\partial X} = 2\pi\chi(X),$$

where k_g is the *geodesic curvature* of ∂X.

23.3 Invariance of de Rham Cohomology

Finally, we briefly touch on the deep relationship between the theories of de Rham cohomology and singular homology. It turns out that although differential forms are a local geometric construction, the de Rham cohomology structure is actually a purely topological invariant that is preserved under diffeomorphisms. This is illustrated by a link between de Rham cohomology and *singular homology*, which is topological. We'll discuss this more in the next lecture.

24 April 26th, 2021

Today we continue discussing de Rham cohomology, in particular, its connections to other homology and cohomology theories from topology. An illustrated video explanation of some of this lecture's material can be found at https://youtu.be/2ptFnIj71SM.

24.1 Recap of Cohomology

Let's review what we mean when we talk about de Rham cohomology. Recall that the space of all k-forms on a differentiable manifold X is denoted $A^k(X)$. This is a very large, infinite-dimensional vector space (Hilbert space). Furthermore, there are two subspaces of closed and exact forms denoted $Z^k(X)$ and $B^k(X)$, respectively. The k-th de Rham cohomology group is $H^k_{\text{dR}}(X) = Z^k(X)/B^k(X)$, the quotient of these vector spaces. It can be shown that $X \rightsquigarrow H^k_{\text{dR}}(X)$ is a contravariant functor from compact differentiable manifolds to vector spaces on the reals, which is also a subcategory in the category of abelian groups **Ab**.

A very interesting fact about de Rham cohomology is that under mild conditions on the behavior of X, the k-th de Rham cohomology group of X is finite-dimensional. This means that even though the vector spaces of differential forms (as well as closed and exact forms) are far too large to study, being function spaces, the de Rham cohomology groups are actually much more manageable. There is a deep connection between the topology of spaces and cohomology:

- If $\pi_1(X) = 0$, then $H^1_{\text{dR}}(X) = 0$.
- Stokes' theorem tells us that if ω is a closed k-form, and $Y = \partial Z$ for $Z \subset X$ being a $(k+1)$-dimensional submanifold with boundary, then $\int_Y \omega = 0$.
- As a converse, is it true that if $\int_Y \omega = 0$ for all closed k-forms ω, then Y is the boundary of some $(k+1)$-submanifold Z?

It turns out that the answer to this last question is negative, but exploring this question is precisely what motivates the theory of singular homology.

24.2 Singular Homology and de Rham's Theorem

Motivated by our previous example, we naively try to define some kind of object to formalize our boundaries. Given a compact manifold X, consider the set of all submanifolds $Y \subset X$, under the equivalence relation that

$$Y \sim Y' \iff Y - Y' = \partial Z,$$

where $Y - Y'$ denotes the signed difference above, and Z is some submanifold with boundary. Then, the set of equivalence classes of this relation provides us the distinct submanifolds of X based on not being boundaries, or in other words, how many "holes" they have in them. There are a few problems with this naive approach, though:

- This is not a nice algebraic object in any way; it is merely a set without additional structure.
- It is very hard to calculate this equivalence relation due to its size, which makes it not very useful. Typically, we won't gain any more information that we don't already know.
- Counting submanifolds is just way too complex.

To remedy this, we use the fact that manifolds can be *triangulated* by simplices, which turns this into a discrete problem. This motivates the theories of simplicial and singular homology.

Definition 24.1 (k-simplex)**.** A *k-simplex* Δ^k is the convex hull of $k+1$ linearly independent points in $\mathbb{R}^k$. This has $k+1$ vertices and $k+1$ faces, each of which is isomorphic to Δ^{k-1}.

For example, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. When we say that manifolds are *triangulable*, we mean that we can write any compact k-manifold X as homeomorphic to a union of k-simplices $\bigcup_\alpha \Delta_\alpha^k$, such that any pair of simplices is either disjoint or intersects at an ℓ-simplex $\Delta_\alpha^k \cap \Delta_\beta^k$, where $\ell < k$. For example, two triangles in a polytope could intersect at an edge Δ_1 or corner Δ_0.

Now, we slightly abuse notation and give each Δ_α^k a differential structure, even though it has "corners" when $k \geq 2$, which is not allowed under our standard definition of manifolds with boundary. The differential structure is given by the containing space $\mathbb{R}^{k+1}$. Then, any function $f : X \to Y$ can be factored into a set of smooth functions $f_\alpha : \Delta_\alpha^k \to Y$ for each simplex.[22]

Definition 24.2 (Singular k-chain)**.** Given a topological space X, we say that a *singular k-simplex* is a map $\sigma : \Delta^k \to X$. Then, a *singular k-chain* is an element of $C_k(X)$, the free abelian group generated by sums and differences of singular k-simplices on a topological space.

Definition 24.3 (Boundary operator)**.** Given a singular k-simplex $f \in C_k(X)$, we say that the boundary of f is a singular $(k-1)$-chain $\partial f \in C_{k-1}(X)$ given by

$$\partial f = \sum_i \pm f|_{\Delta_i^k},$$

where the sign is based on the orientation of the boundaries of each face $\Delta_i^k \subset \partial \Delta^k$. This extends to a linear map $\partial : C_k(X) \to C_{k-1}(X)$, which is called the *boundary operator*.

By a basic combinatorial argument, $\partial^2 = 0$. This means that we have maps

$$\cdots \to C_{k+1}(X) \xrightarrow{\partial_{k+1}} C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \to \cdots,$$

which are the *singular chain complex* corresponding to X. This is the reverse of our *de Rham cochain complex* construction, which was based on the exterior derivative $\mathrm{d} : A^k(X) \to A^{k+1}(X)$.

Definition 24.4 (Singular homology)**.** We define the k-th subspace of *cycles* by $Z_k(X) = \ker \partial_k$, and the subspace of *boundaries* by $B_k(X) = \operatorname{im} \partial_{k+1}$. Then, the *$k$-th singular homology group* of X is the quotient $H_k(X, \mathbb{R}) = Z_k(X)/B_k(X)$.

A crucial observation is that even though simplices are not manifolds with boundary, a generalized form of Stokes' theorem still applies to simplices by simply taking their signed boundary operator. In particular, if ω is a $(k-1)$-form on Δ^k, then

$$\int_{\Delta^k} \mathrm{d}\omega = \sum_i \pm \int_{\Delta_i^k} \omega.$$

Given a closed form on X and a collection of simplices of X whose boundary sums to zero, we can integrate that form over the simplices to produce a real number. By Stokes' theorem, this is the same if we add an exact form to ω or a boundary to the simplices, so we have a well-defined integration map $H_{\mathrm{dR}}^k(X) \times H_k(X, \mathbb{R}) \to \mathbb{R}$. In general, linear maps can be either degenerate (having nontrivial kernel) or non-degenerate.

[22] As a technical point, we need $\mathcal{C}^2$ functions in the arguments to follow, but this doesn't matter because topological singular homology is the same as smooth singular homology by the Whitney approximation theorem.

Theorem 24.5 (De Rham)**.** *The integration map is non-degenerate, i.e.,* $H^k_{\mathrm{dR}}(X) = H_k(X, \mathbb{R})^*$.

Hence, we have a natural isomorphism between the corresponding de Rham cohomology groups and singular homology groups of a compact differentiable manifold X. In particular, this illustrates as a consequence the extremely nontrivial fact that de Rham cohomology is a topological invariant!

Corollary 24.5.1. *If* X *and* Y *are differentiable manifolds, and* $\phi : X \to Y$ *is a homeomorphism, then* $H^k_{\mathrm{dR}}(X) = H^k_{\mathrm{dR}}(Y)$.

24.3 Poincaré Duality

Recall the formula $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta \pm \omega \wedge \mathrm{d}\eta$. This has the following consequences:

- If ω and η are both closed, then $\omega \wedge \eta$ is closed.
- If ω is closed and η is exact, then $\omega \wedge \eta$ is exact.

Therefore, we have a bilinear map $H^k_{\mathrm{dR}}(X) \times H^\ell_{\mathrm{dR}}(X) \to H^{k+\ell}_{\mathrm{dR}}(X)$. Furthermore, note that if $n = \dim X$, then the n-th de Rham cohomology of X is isomorphic to $\mathbb{R}$, as it only consists of some scalar multiples of the volume form.

Theorem 24.6 (Poincaré duality)**.** *As noted above, observe that the wedge product gives us an induced pairing* $\wedge : H^k_{\mathrm{dR}}(X) \times H^{n-k}_{\mathrm{dR}}(X) \to H^n_{\mathrm{dR}}(X)$, *for any* $0 \le k \le n$. *The Poincaré duality theorem states that this pairing is nondegenerate, which induces an isomorphism between the de Rham cohomology group of order* k *and the dual of the corresponding group of order* $n - k$.

Corollary 24.6.1. *By Poincaré duality and de Rham's theorem, we have an isomorphism diagram*

$$\begin{array}{ccc} H^{n-k}_{\mathrm{dR}}(X) & = & H^k_{\mathrm{dR}}(X)^* \\ \| & & \| \\ H_{n-k}(X, \mathbb{R})^* & = & H_k(X, \mathbb{R}). \end{array}$$

This induces a non-degenerate bilinear map $H_k(X, \mathbb{R}) \times H_{n-k}(X, \mathbb{R}) \to \mathbb{R}$. *If* $[Y] \in H_k(X, \mathbb{R})$ *and* $[Z] \in H_{n-k}(X, \mathbb{R})$ *are the homeomorphism equivalence classes of two compact oriented submanifolds* $Y, Z \subset X$ *represented as simplicial chains of dimension* k *and* $n - k$, *then this map sends* $([Y], [Z])$ *to the oriented intersection number* $I(Y, Z)$.

This material would usually be covered in the first few weeks of Math 231a, graduate-level algebraic topology. Joe hopes to give us a taste of how much interesting algebraic content is within reach after learning the tools of differential topology.

25 April 28th, 2021

This is the last lecture of the course. We discuss the modern definition of manifolds, give more important examples, and provide an overview of some useful additional structures on manifolds.

25.1 Historical Development

First, let's talk about the hstorical development of group theory. In the 19^{th} century, a *group* was defined as a subset of GL_n closed under matrix multiplication and inversion. These are called *finite-dimensional faithful representations* of a group, in the terminology of modern representation theory. All finite groups can be represented in this way.

In the 20^{th} century, there was a fundamental abstract shift. The new definition of a group, which we see in algebra classes today, is a set G with the additional structure of a *law of composition* $G \times G \to G$, which is associative, has an inverse, and has an identity element.

Analogously, there was a fundamental shift in the definition of a manifold. In the 19^{th} century, a manifold was seen as a subset of $\mathbb{R}^N$ that is locally defined by independent $\mathcal{C}^\infty$ functions, i.e., something that is locally diffeomorphic to Euclidean space. This is the (outdated) definition that we primarily used in this course, as it is concrete and easy to understand. However, in the 20^{th} century, there was a shift in perspective that moved away from Euclidean embeddings.

Definition 25.1 (Atlas)**.** On a topological space X, an *atlas* is an open cover $X = \bigcup_\alpha U_\alpha$ with a collection of homeomorphisms $\phi_\alpha : U_\alpha \to \mathbb{R}^n$, such that for any two sets U_α, U_β with nonzero intersection in the open cover, the *transition maps* $\phi_\alpha \circ \phi_\beta^{-1}$ are $\mathcal{C}^\infty$ where defined.

This definition does not require that we define smooth functions on the manifold itself, just smooth functions of the form $\mathbb{R}^n \to \mathbb{R}^n$. We say that two atlases are *equivalent* if their atlases have compatible transition maps. Then, the modern definition of a smooth manifold is a topological space X, along with a specified equivalence class of atlases.

Note. It is not enough to simply give a topological space X and infer the differential structure. This is because atlases are not unique; for example, $\mathbb{R}^4$ has uncountably many non-diffeomorphic structures as a differential manifold!

25.2 Germs and Sheaves

The atlas definition of a manifold was introduced in the early 20^{th} century, and it became ubiquitous in the later half of the century. However, now that we are in the 21^{th} century, Joe Harris prefers a slightly different definition of manifold. Instead of specifying an equivalence class of atlases and deriving smooth functions from them, **we want to specify the structure in terms of which continuous functions are differentiable**.

Definition 25.2 (Sheaf-theoretic manifold)**.** A $\mathcal{C}^\infty$ manifold X of dimension n is a topological space X, where for all open sets $U \subset X$, we have a subring

$$\mathcal{C}_X^\infty(U) \subset \mathcal{C}_X(U),$$

where $\mathcal{C}_X(U)$ is the ring of continuous functions and $\mathcal{C}_X^\infty(U)$ is the ring of smooth functions, such that this structure is locally isomorphic to $\mathbb{R}^n$.

Another equivalent definition of a manifold is formulated in terms of *germs*, which are a very popular concept in topology for defining structures through sheaves. The problem with local functions f and g on a manifold is that you can't add them together when they are defined on different open subsets. Instead, the solution is to fix some $p \in X$ and consider

$$\{(U, f) : p \in U, f : U \to \mathbb{R}\}.$$

In this collection of continuous functions f that are defined on open neighborhoods of p, we say that $(U, f) \sim (V, g)$ if there exists an open set $W \subset U \cap V$ such that $p \in W$, and $f|_W = g|_W$. Under this equivalence relation, the set has the structure of an algebraic ring. This is called the *ring of germs* around a point p, and it is denoted $\mathcal{C}_p(X)$.

Definition 25.3 (Germ-theoretic manifold)**.** A topological space X is called a smooth manifold when endowed with a subring of the ring of germs $\mathcal{C}_p^\infty(X) \subset \mathcal{C}_p(X)$, for each $p \in X$.

We won't do too much with these 21^{th} century definitions, but Joe Harris notes that if we take further courses or do additional reading in differential topology, this notation is very common, so it's useful to be exposed to it early on.

25.3 Grassmannians

Let's give one popular example where the atlas definition is a much more convenient way to prove that something is a manifold, and embedding is trickier.

Definition 25.4 (Grassmannian)**.** The *Grassmannian* $G(k, n)$ is the locus of all k-dimensional linear subspaces Λ of $\mathbb{R}^n$.

For example, $G(1, n) = \mathbb{RP}^{n-1}$ by equivalence of definition. In general, we can give $G(k, n)$ the structure of a $\mathcal{C}^\infty$ manifold by introducing coordinate charts. For example, we can do this for $\mathbb{RP}^{n-1}$, which has dimension $n-1$, by fixing a hyperplane $H \subset \mathbb{R}^n$ and shifting it by its normal vector (affine transformation), which produces a coordinate chart for almost all directions in $\mathbb{RP}^{n-1}$, except those directions parallel to H. Let's generalize this argument.

Proposition 25.5. *$G(k, n)$ is a smooth manifold of dimension $k(n-k)$.*

Proof. To represent the Grassmannian, we consider the row-space of some $k \times n$ matrix of rank k. However, these do not necessarily represent distinct linear subspaces, as left-multiplying by some invertible matrix in GL_k preserves the row-space of the matrix. This already gives $G(k, n)$ the topological structure of a quotient space

$$G(k,n) = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix} \text{ of rank } k \right\} \Big/ \ \mathrm{GL}_k \ \text{ acting on the left.}$$

To define coordinate charts, we first fix a linear subspace $\Gamma \subset \mathbb{R}^n$ of dimension $n-k$. Then, let

$$U_\Gamma = \{\Lambda \in G(k, n) : \Lambda \cap \Gamma = 0\}.$$

We claim that U_Γ is homeomorphic to $\mathbb{R}^{k(n-k)}$, and this coordinate chart is globally smooth. The homeomorphism is defined as follows. Without loss of generality, assume we take some coordinate basis such that $\Gamma = \{x \in \mathbb{R}^n \mid x_1 = \cdots = x_k = 0\}$. Then,

$$\Lambda \cap \Gamma = \{0\} \iff \Lambda \sim \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix} = A,$$

where the first $k \times k$ matrix minor of A_0 is nonsingular. Since A is only unique up to left multiplication by a matrix in GL_k, there is a unique matrix representative of Λ with the block form (I_k, B), and B is a matrix of dimensions $k \times (n-k)$. This provides a bijection $U_\Lambda \cong \mathbb{R}^{k(n-k)}$. □

Exercise 25.1. Show that the coordinate charts described above form a smooth atlas by writing down the transition maps explicitly.

Finally, just for comparison, we would like to show one more definition of the Grassmannian in terms of embedding in Euclidean space. Suppose that $\Lambda \subset \mathbb{R}^n$ is a linear subspace of dimension k, and let $v_1, \ldots, v_k$ form a basis for Λ. Then, the wedge product $v_1 \wedge \cdots \wedge v_k$ is a k-form on Λ, and it is the same up to scalar multiplication no matter which basis is chosen. Therefore, this provides an embedding[23]

$$\bigwedge^k \Lambda \subset \bigwedge^k \mathbb{R}^n,$$

which is called the *Plucker embedding.* Then, embedding the Grassmannian in Euclidean space can be reduced to embedding $\bigwedge^k \mathbb{R}^n$ in Euclidean space, which we have seen already in the homework problems for this class.

25.4 Metric Structure on Manifolds

Suppose that X is a manifold of dimension n, in any of our definitions above. We would like to introduce some other useful geometric structures.

Definition 25.6 (Riemannian metric)**.** A *Riemannian metric* on X is a positive definite symmetric bilinear form (i.e., inner product) on the tangent space T_pX, for each $p \in X$, which is smooth. In terms of local coordinates $x_1, \ldots, x_n$ in X, we can write the inner product as

$$\sum_{i,j} a_{ij}\, \mathrm{d}x_i\, \mathrm{d}x_j,$$

where the functions a_{ij} are smoothly varying.

We call a manifold X with specified Riemannian metric g a *Riemannian manifold.* Analogously, if we require that the bilinear form be skew-symmetric, then we call this a *symplectic manifold.* These definitions have some differences. All differentiable manifolds can be given a Riemannian metric, but not all manifolds admit a symplectic structure.

Proposition 25.7. *The Riemannian metric g induces a metric on the manifold X.*

Proof. We define the length of a path $f : I \to X$ by the integral of its arc length, as defined by the Riemannian metric. Then, the *geodesic distance* between p and q is the shortest length of a path between those two points. □

Riemannian manifolds are very common in physics (relativity involves pseudo-Riemannian manifolds), geometry processing (geodesics, curvature, etc.), and other fields of mathematics. In particular, the Guass-Bonnet theorem as stated previously involves implicit Riemannian structure. Intuitively, if differential manifolds are made of rubber (bent and stretched), then Riemannian manifolds are made of paper, which can only be deformed isometrically. For more on this topic, Math 136 covers differential geometry in much greater depth.

[23] A funny quote from Joe Harris in lecture: "Oh god, why did I use Lambdas? Okay, Lambdas from now on are going to all have feet, to distinguish them from wedges."

References

[GP10] Victor Guillemin and Alan Pollack. *Differential Topology*, volume 370. American Mathematical Soc., 2010.